

## Types of Integrals

### 1 Mark Questions

1. Find  $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ .

Delhi 2014C

$$\begin{aligned} \text{Let } I &= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx - \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \sec^2 x dx - \int \operatorname{cosec}^2 x dx \\ &= \tan x + \cot x + C \end{aligned} \quad (1)$$

2. Find  $\int \frac{\sin^6 x}{\cos^8 x} dx$ .

All India 2014C



$$\text{Let } I = \int \frac{\sin^6 x}{\cos^8 x} dx = \int \tan^6 x \sec^2 x dx$$

$$\text{Put } \tan x = t$$

$$\Rightarrow \sec^2 x dx = dt$$

$$\therefore I = \int t^6 dt = \frac{t^7}{7} + C = \frac{\tan^7 x}{7} + C \quad (1)$$

3. Evaluate  $\int \frac{dx}{\sin^2 x \cos^2 x}$ .

Delhi 2014C; Foreign 2014

💡 Firstly, divide numerator and denominator by  $\cos^4 x$  and use  $\sec^2 x = 1 + \tan^2 x$ , then put  $\tan x = t$  and integrate.

$$\text{Let } I = \int \frac{dx}{\sin^2 x \cos^2 x}$$

On dividing the numerator and denominator by  $\cos^4 x$ , we get

$$I = \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^2 x} dx$$

$$\Rightarrow I = \int \frac{(1 + \tan^2 x) \cdot \sec^2 x}{\tan^2 x} dx$$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\therefore I = \int \frac{1+t^2}{t^2} dt = \int 1 dt + \int \frac{1}{t^2} dt$$

$$\Rightarrow I = t - \frac{1}{t} + C$$

$$\Rightarrow I = \tan x - \cot x + C \quad (1)$$

4. Evaluate  $\int \cos^{-1}(\sin x) dx$ .

Delhi 2014C

$$\begin{aligned}
 \text{Let } I &= \int \cos^{-1}(\sin x) dx \\
 &= \int \cos^{-1} \left[ \cos \left( \frac{\pi}{2} - x \right) \right] dx \\
 &= \int \left( \frac{\pi}{2} - x \right) dx \quad [\because \cos^{-1}(\cos \theta) = \theta] \\
 &= \frac{\pi}{2} \int dx - \int x dx = \frac{\pi}{2} x - \frac{x^2}{2} + C
 \end{aligned}$$

5. Write the anti-derivative of  $\left( 3\sqrt{x} + \frac{1}{\sqrt{x}} \right)$ .

Delhi 2014

$$\begin{aligned}
 \text{Anti-derivative of } \left( 3\sqrt{x} + \frac{1}{\sqrt{x}} \right) \\
 &= \int \left( 3\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx = 3 \int \sqrt{x} dx + \int \frac{1}{\sqrt{x}} dx \\
 &= 3 \left( \frac{x^{1/2+1}}{1/2+1} \right) + \left[ \frac{x^{-1/2+1}}{-1/2+1} \right] + C \\
 &= 2(x^{3/2} + x^{1/2}) + C \quad (1)
 \end{aligned}$$

6. Evaluate  $\int (1-x)\sqrt{x} dx$ .


HOTS; Delhi 2012

💡 Firstly, multiply the two functions and then use  
 $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1.$

$$\begin{aligned}
 \text{Let } I &= \int (1-x)\sqrt{x} dx = \int (\sqrt{x} - x\sqrt{x}) dx \\
 &= \int (x^{1/2} - x^{3/2}) dx = \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} + C \\
 &\quad \left[ \because \int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1 \right] (1)
 \end{aligned}$$

7. Given,  $\int e^x (\tan x + 1) \sec x dx = e^x f(x) + C$ . Write  $f(x)$  satisfying above.

All India 2012; Foreign 2011

 Use the relation  $\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$  and simplify it.

Given that,  $\int e^x (\tan x + 1) \sec x dx = e^x \cdot f(x) + C$

$$\Rightarrow \int e^x (\sec x + \sec x \tan x) dx = e^x f(x) + C$$

$$\Rightarrow e^x \cdot \sec x + C = e^x f(x) + C$$

$$\left[ \begin{array}{l} \because \int e^x \{f(x) + f'(x)\} dx = e^x f(x) \\ \text{and } \frac{d}{dx} (\sec x) = \sec x \tan x \end{array} \right]$$

On comparing both sides, we get

$$f(x) = \sec x \tag{1}$$

8. Evaluate  $\int \frac{2}{1 + \cos 2x} dx$ .

Foreign 2012

Let  $I = \int \frac{2}{1 + \cos 2x} dx$

$$= \int \frac{2}{2 \cos^2 x} dx \quad [\because \cos 2\theta = 2 \cos^2 \theta - 1]$$

$$= \int \sec^2 x dx = \tan x + C \tag{1}$$

9. Write the value of  $\int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx$ . All India 2012C

$$\text{Let } I = \int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx$$

$$\text{Put } 3x^2 + \sin 6x = t$$

$$\Rightarrow 6x + 6 \cos 6x = \frac{dt}{dx} \Rightarrow (x + \cos 6x) dx = \frac{dt}{6}$$

$$\therefore I = \int \frac{dt}{6t} = \frac{1}{6} \log|t| + C \quad \left[ \because \int \frac{1}{x} dx = \log|x| \right]$$

$$= \frac{1}{6} [\log|(3x^2 + \sin 6x)|] + C \quad (1)$$

**10.** Write the value of  $\int \frac{dx}{x^2 + 16}$ .

Delhi 2011

$$\text{Let } I = \int \frac{dx}{x^2 + 16} = \int \frac{dx}{x^2 + (4)^2}$$

$$= \frac{1}{4} \tan^{-1} \frac{x}{4} + C$$

$$\left[ \because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \right] (1)$$

**11.** Write the value of  $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$ .

Delhi 2012C, 2011

$$\text{Let } I = \int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx = \int \frac{\left( \frac{1}{\cos^2 x} \right)}{\left( \frac{1}{\sin^2 x} \right)} dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x} dx$$

$$= \int \tan^2 x dx = \int (\sec^2 x - 1) dx$$

$$[\because \tan^2 x + 1 = \sec^2 x]$$

$$= \int \sec^2 x dx - \int 1 dx = \tan x - x + C$$

$$[\because \int \sec^2 x dx = \tan x] (1)$$

**12.** Write the value of  $\int \frac{2 - 3 \sin x}{\cos^2 x} dx$ . Delhi 2011


$$\begin{aligned} \text{Let } I &= \int \frac{2 - 3 \sin x}{\cos^2 x} dx = \int \left( \frac{2}{\cos^2 x} - \frac{3 \sin x}{\cos^2 x} \right) dx \\ &= \int (2 \sec^2 x - 3 \sec x \tan x) dx \\ &= 2 \int \sec^2 x dx - 3 \int \sec x \tan x dx \\ &= 2 \tan x - 3 \sec x + C \quad (1) \end{aligned}$$

$$\left[ \begin{array}{l} \because \int \sec^2 x dx = \tan x \\ \text{and } \int \sec x \tan x dx = \sec x \end{array} \right]$$

**13.** Evaluate  $\int \frac{dx}{\sqrt{1-x^2}}$ . All India 2011

$$\begin{aligned} \text{Let } I &= \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dx}{\sqrt{(1)^2 - x^2}} = \sin^{-1} x + C \\ &\left[ \because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \right] (1) \end{aligned}$$

**14.** Evaluate  $\int \frac{(\log x)^2}{x} dx$ . All India 2011

 The differentiation of  $\log x$  is in denominator. So, firstly put  $\log x = t$  and adjust the integral in terms of  $t$  and then integrate.

$$\text{Let } I = \int \frac{(\log x)^2}{x} dx$$

$$\text{Put } \log x = t \Rightarrow \frac{1}{x} dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{(\log x)^2}{x} dx = \int t^2 dt \\ &= \frac{t^3}{3} + C = \frac{(\log x)^3}{3} + C \quad (1) \end{aligned}$$



**15.** Evaluate  $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$ . **All India 2011**

Let  $I = \int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

Put  $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$$\begin{aligned} \therefore I &= \int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^t dt \\ &= e^t + C \quad [ \because \int e^x dx = e^x ] \\ &= e^{\tan^{-1} x} + C \end{aligned} \quad (1)$$

**16.** Evaluate  $\int (ax+b)^3 dx$ . **All India 2011**

Let  $I = \int (ax+b)^3 dx$

Put  $t = ax+b$

$$\Rightarrow \frac{dt}{dx} = a \Rightarrow \frac{dt}{a} = dx$$

$$\begin{aligned} \therefore I &= \int \frac{t^3}{a} dt = \frac{1}{a} \cdot \frac{t^4}{4} + C \\ &= \frac{(ax+b)^4}{4a} + C \end{aligned} \quad (1)$$

**17.** Evaluate  $\int \frac{(1+\log x)^2}{x} dx$ .

**Foreign 2011; Delhi 2009**

Let  $I = \int \frac{(1+\log x)^2}{x} dx$

Put  $1+\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\begin{aligned} \therefore I &= \int \frac{(1+\log x)^2}{x} dx = \int t^2 dt = \frac{t^3}{3} + C \\ &= \frac{(1+\log x)^3}{3} + C \end{aligned} \quad (1)$$

**18.** Evaluate  $\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$ .

Foreign 2011

Let  $I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$

Put  $e^{2x} + e^{-2x} = t$

$$\Rightarrow (2e^{2x} - 2e^{-2x}) dx = dt \quad \left[ \because \frac{d}{dx} (e^{ax}) = ae^{ax} \right]$$

$$\Rightarrow (e^{2x} - e^{-2x}) dx = \frac{dt}{2}$$

$$\therefore I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx = \frac{1}{2} \int \frac{dt}{t}$$

$$= \frac{1}{2} \log |t| + C$$

$$= \frac{1}{2} \log |e^{2x} + e^{-2x}| + C \quad (1)$$

**19.** Evaluate  $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ .

Foreign 2011

Let  $I = \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

Put  $\sqrt{x} = t$

$$\Rightarrow \frac{1}{2\sqrt{x}} dx = dt \Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$$

$$\therefore I = \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = 2 \int \cos t dt$$

$$= 2 \sin t + C = 2 \sin \sqrt{x} + C \quad (1)$$

**20.** Evaluate  $\int \frac{2 \cos x}{\sin^2 x} dx$ .


All India 2011C, 2009, 2008



$$\begin{aligned} \text{Let } I &= \int \frac{2 \cos x}{\sin^2 x} dx = \int 2 \operatorname{cosec} x \cot x dx \\ &= -2 \operatorname{cosec} x + C \end{aligned} \quad (1)$$

[ $\because \int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x$ ]

**21.** Evaluate  $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$ . Delhi 2011C

 Firstly, factorise numerator and cancel out common from numerator and denominator and then integrate.

$$\begin{aligned} \text{Let } I &= \int \frac{x^3 - x^2 + x - 1}{x - 1} dx \\ &= \int \frac{x^2(x - 1) + 1(x - 1)}{x - 1} dx \\ &= \int \frac{(x^2 + 1)(x - 1)}{x - 1} dx = \int (x^2 + 1) dx \\ &= \frac{x^3}{3} + x + C \end{aligned} \quad (1)$$

**22.** Write the value of  $\int \frac{1 - \sin x}{\cos^2 x} dx$ . All India 2011C

$$\begin{aligned} \text{Let } I &= \int \frac{1 - \sin x}{\cos^2 x} dx = \int \left( \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx \\ &= \int \sec^2 x dx - \int \sec x \tan x dx \\ &= \tan x - \sec x + C \end{aligned} \quad (1)$$

**23.** Evaluate  $\int \frac{2 \cos x}{3 \sin^2 x} dx$ . All India 2011C

$$\text{Let } I = \int \frac{2 \cos x}{3 \sin^2 x} dx$$

$$\text{Let } \sin x = t \Rightarrow \cos x dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{2dt}{3t^2} = \frac{2}{3} \int \frac{dt}{t^2} = \frac{2}{3} \frac{t^{-1}}{(-1)} \\ &= \frac{-2}{3} (\sin x)^{-1} + C = \frac{-2}{3 \sin x} + C \quad (1) \end{aligned}$$

### Alternate Method

$$\begin{aligned} \text{Let } I &= \int \frac{2 \cos x}{3 \sin^2 x} dx = \frac{2}{3} \int \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} dx \\ &= \frac{2}{3} \int \operatorname{cosec} x \cot x dx \\ &= \frac{2}{3} (-\operatorname{cosec} x) + C = -\frac{2}{3 \sin x} + C \quad (1) \end{aligned}$$

**24.** Evaluate  $\int \frac{x^3 - 1}{x^2} dx$ .

Delhi 2010C

$$\begin{aligned} \text{Let } I &= \int \frac{x^3 - 1}{x^2} dx = \int \left( \frac{x^3}{x^2} - \frac{1}{x^2} \right) dx \\ &= \int x dx - \int \frac{1}{x^2} dx = \frac{x^2}{2} - \frac{x^{-1}}{-1} + C \\ &= \frac{x^2}{2} + \frac{1}{x} + C \quad (1) \end{aligned}$$

**25.** Evaluate  $\int \sec^2 (7 - 4x) dx$ .

Delhi 2010; All India 2010

$$\begin{aligned} \text{Let } I &= \int \sec^2 (7 - 4x) dx \\ &= \frac{\tan (7 - 4x)}{-4} + C \\ &\quad \left[ \because \int \sec^2 ax dx = \frac{\tan ax}{a} \right] \\ &= -\frac{\tan (7 - 4x)}{4} + C \quad (1) \end{aligned}$$

**26.** Evaluate  $\int \frac{\log x}{x} dx$ .

All India 2010C

$$\text{Let } I = \int \frac{\log x}{x} dx$$

$$\text{Put } \log x = t \Rightarrow \frac{1}{x} dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{\log x}{x} dx = \int t dt = \frac{t^2}{2} + C \\ &= \frac{(\log x)^2}{2} + C \end{aligned} \quad (1)$$

**27.** Evaluate  $\int 2^x dx$ .

All India 2010C

$$\text{Let } I = \int 2^x dx$$

$$= \frac{2^x}{\log 2} + C \quad \left[ \because \int a^x dx = \frac{a^x}{\log a} \right]$$

**Alternate Method**

$$\text{Let } I = \int 2^x dx$$

$$\text{Again, let } 2^x = t$$

On taking log both sides, we get

$$x \log 2 = \log t$$

$$\Rightarrow \log 2 dx = \frac{1}{t} dt \Rightarrow dx = \frac{1}{t \log 2} dt$$

$$\begin{aligned} \therefore I &= \int \frac{t dt}{t \log 2} = \frac{1}{\log 2} \int dt \\ &= \frac{t}{\log 2} + C = \frac{2^x}{\log 2} + C \end{aligned} \quad (1)$$

**28.** Evaluate  $\int \frac{\log(\sin x)}{\tan x} dx$ .

HOTS; All India 2009C

💡 The differentiation of  $\log(\sin x)$  is  $\tan x$ , which exists as denominator. So, solve it by substitution method.

Let 
$$I = \int \frac{\log(\sin x)}{\tan x} dx$$

Put  $\log(\sin x) = t$

$\Rightarrow \frac{1}{\sin x} \cdot \cos x dx = dt \Rightarrow \cot x dx = dt$

$\Rightarrow \frac{1}{\tan x} dx = dt \left[ \because \cot x = \frac{1}{\tan x} \right]$

$\therefore I = \int \frac{\log(\sin x)}{\tan x} dx = \int t dt$

$= \frac{t^2}{2} + C = \frac{(\log \sin x)^2}{2} + C \quad (1)$

**29.** Evaluate  $\int (\operatorname{cosec}^2 x - \cot x) e^x dx$ . **Delhi 2009C**

Let  $I = \int (\operatorname{cosec}^2 x - \cot x) e^x dx$

$= - \int (\cot x - \operatorname{cosec}^2 x) e^x dx$

$= - \int [\cot x + (-\operatorname{cosec}^2 x)] e^x dx$

$= - e^x \cot x + C$

$[\because \int e^x \{f(x) + f'(x)\} dx = e^x \cdot f(x)] \quad (1)$

**30.** Evaluate  $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx$ .

**All India 2009**

Do same as Que 19. [**Ans.**  $2 \tan \sqrt{x} + C$ ]

**31.** Evaluate  $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$ .

**All India 2009**

Do same as Que 19. [**Ans.**  $-2 \cos \sqrt{x} + C$ ]

**32.** Evaluate  $\int \frac{\sec^2 x}{3 + \tan^2 x} dx$ .

**Delhi 2009**

$$\text{Let } I = \int \frac{\sec^2 x}{3 + \tan^2 x} dx$$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{\sec^2 x}{3 + \tan^2 x} dx = \int \frac{dt}{3 + t^2} = \int \frac{dt}{(\sqrt{3})^2 + t^2} \\ &= \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{t}{\sqrt{3}} \right) + C \left[ \because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \right] \\ &= \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{\tan x}{\sqrt{3}} \right) + C \end{aligned} \quad (1)$$

**33.** Evaluate  $\int \frac{x^2}{1+x^3} dx$ .

Delhi 2009

$$\text{Let } I = \int \frac{x^2}{1+x^3} dx$$

$$\text{Put } 1+x^3 = t \Rightarrow 3x^2 dx = dt$$

$$\Rightarrow x^2 dx = \frac{dt}{3}$$

$$\begin{aligned} \therefore I &= \int \frac{x^2}{1+x^3} dx = \frac{1}{3} \int \frac{dt}{t} \\ &= \frac{1}{3} \log |t| + C \quad \left[ \because \int \frac{dx}{x} = \log |x| \right] \\ &= \frac{1}{3} \log |1+x^3| + C \end{aligned} \quad (1)$$

**34.** Evaluate  $\int \sin^3 x dx$ .

HOTS; All India 2009

$$\text{Let } I = \int \sin^3 x dx = \int \frac{3 \sin x - \sin 3x}{4} dx$$

$$= \frac{1}{4} \int 3 \sin x dx - \frac{1}{4} \int \sin 3x dx$$

$$= \frac{1}{4} \left( -3 \cos x + \frac{\cos 3x}{3} \right) + C \quad (1)$$

#### 4 Mark Questions

**35.** Evaluate  $\int \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}} dx$ .

All India 2009

$$\text{Let } I = \int \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}} dx = \int \sqrt{\frac{2 \sin^2 x}{2 \cos^2 x}} dx$$

$$\left[ \begin{array}{l} \because \cos 2\theta = 1 - 2 \sin^2 \theta \\ \qquad \qquad = 2 \cos^2 \theta - 1 \end{array} \right]$$

$$= \int \tan x dx = \log(\sec x) + C \quad (1)$$

**36.** Evaluate  $\int (x - 3) \sqrt{x^2 + 3x - 18} dx$ . Delhi 2014

? Here, integral is of the form  $(px - q)\sqrt{ax^2 + bx + c}$ , so firstly write  $x - 3$  as  $x - 3 = A \frac{d}{dx}(x^2 + 3x - 18) + B$  and find  $A$  and  $B$ . Then, integrate by using suitable method.

$$\text{Let } I = \int (x - 3) \sqrt{x^2 + 3x - 18} dx$$

Here, we can write  $(x - 3)$  as

$$x - 3 = A \frac{d}{dx}(x^2 + 3x - 18) + B$$

$$\Rightarrow x - 3 = A(2x + 3) + B$$

On equating the coefficients of  $x$  and constant term from both sides, we get

$$2A = 1 \text{ and } 3A + B = -3$$

$$\Rightarrow A = \frac{1}{2} \text{ and } 3 \times \frac{1}{2} + B = -3$$

$$\Rightarrow A = \frac{1}{2} \text{ and } B = -\frac{3}{2} - 3$$

$$\Rightarrow A = \frac{1}{2} \text{ and } B = -\frac{9}{2} \quad (1)$$

Thus, the given integral reduces in the following form

$$\begin{aligned}
 I &= \int \left\{ \frac{1}{2}(2x+3) - \frac{9}{2} \right\} \sqrt{x^2+3x-18} \, dx \\
 \Rightarrow I &= \frac{1}{2} \int (2x+3) \sqrt{x^2+3x-18} \, dx \\
 &\quad - \frac{9}{2} \int \sqrt{x^2+3x-18} \, dx \\
 &= \frac{1}{2} I_1 - \frac{9}{2} I_2 \quad \dots(i)
 \end{aligned}$$

where,  $I_1 = \int (2x+3) \sqrt{x^2+3x-18} \, dx$

Put  $x^2+3x-18 = t$

$\Rightarrow (2x+3) \, dx = dt$

$\therefore I_1 = \int t^{1/2} \, dt = \frac{2}{3} t^{3/2} + C_1 \quad (1)$

$$= \frac{2}{3} (x^2+3x-18)^{3/2} + C_1$$

and  $I_2 = \int \sqrt{x^2+3x-18} \, dx$

$$= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - 18 - \frac{9}{4}} \, dx$$

$$= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \frac{81}{4}} \, dx$$

$$= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2} \, dx$$

$$\begin{aligned}
&= \frac{\left(x + \frac{3}{2}\right)}{2} \sqrt{x^2 + 3x - 18} \\
&\quad - \frac{81}{8} \log \left| \left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x - 18} \right| + C_2 \\
&\quad \left[ \because \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} \right. \\
&\quad \quad \left. - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| \right] \\
&= \frac{2x + 3}{4} \sqrt{x^2 + 3x - 18} \\
&\quad - \frac{81}{8} \log \left| \frac{2x + 3}{2} + \sqrt{x^2 + 3x - 18} \right| + C_2 \tag{1}
\end{aligned}$$

On putting the values of  $I_1$  and  $I_2$  in Eq. (i), we get

$$\begin{aligned}
I &= \frac{1}{2} \left[ \frac{2}{3} (x^2 + 3x - 18)^{3/2} + C_1 \right] \\
&\quad - \frac{9}{2} \left[ \frac{2x + 3}{4} \sqrt{x^2 + 3x - 18} \right. \\
&\quad \left. - \frac{81}{8} \log \left| \frac{2x + 3}{2} + \sqrt{x^2 + 3x - 18} \right| + C_2 \right] \\
\Rightarrow I &= \frac{1}{3} (x^3 + 3x - 18)^{3/2} \\
&\quad - \frac{9}{8} (2x + 3) \sqrt{x^2 + 3x - 18} \\
&\quad + \frac{729}{16} \log \left| \frac{2x + 3}{2} + \sqrt{x^2 + 3x - 18} \right| + C \tag{1}
\end{aligned}$$

where,  $C = \frac{C_1}{2} - \frac{9C_2}{2}$

**37.** Evaluate  $\int \frac{x + 2}{\sqrt{x^2 + 5x + 6}} dx$ .

All India 2014





$$\text{Let } I = \int \frac{x+2}{\sqrt{x^2+5x+6}} dx$$

Here,  $(x+2)$  can be written as

$$x+2 = A \frac{d}{dx} (x^2+5x+6) + B$$

$$\Rightarrow x+2 = A(2x+5) + B$$

On equating the coefficients of  $x$  and constant term from both sides, we get

$$2A = 1 \text{ and } 5A + B = 2$$

$$\Rightarrow A = \frac{1}{2} \text{ and then } B = -\frac{1}{2} \quad (1)$$

$$\begin{aligned} \therefore I &= \int \frac{\left\{ \frac{1}{2}(2x+5) - \frac{1}{2} \right\}}{\sqrt{x^2+5x+6}} dx \\ &= \frac{1}{2} \int \frac{2x+5}{\sqrt{x^2+5x+6}} dx \\ &\quad - \frac{1}{2} \int \frac{1}{\sqrt{x^2+5x+6}} dx \end{aligned}$$

$$\Rightarrow I = \frac{1}{2} I_1 - \frac{1}{2} I_2 \quad \dots(i)$$

$$\text{where, } I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}} dx$$

$$\text{Put } x^2+5x+6 = t$$

$$\Rightarrow (2x+5) dx = dt$$

$$\begin{aligned} \therefore I_1 &= \int \frac{1}{\sqrt{t}} dt = 2\sqrt{t} + C_1 \\ &= 2\sqrt{x^2+5x+6} + C_1 \quad \dots(ii) \quad (1) \end{aligned}$$

$$\text{and } I_2 = \int \frac{1}{\sqrt{x^2+5x+6}} dx$$

$$\begin{aligned}
&= \int \frac{1}{\sqrt{x^2 + 2 \times \frac{5}{2} \times x + 6 - \frac{25}{4} + \frac{25}{4}}} dx \\
&= \int \frac{1}{\sqrt{\left(x + \frac{5}{2}\right)^2 + 6 - \frac{25}{4}}} dx \\
&= \int \frac{1}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx \\
&= \log \left| \left(x + \frac{5}{2}\right) + \sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + C_2 \\
&\quad \left[ \because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| \right] \\
\Rightarrow I_2 &= \log \left| x + \frac{5}{2} + \sqrt{x^2 + 5x + 6} \right| + C_2 \dots \text{(iii)} \\
&\qquad\qquad\qquad (1)
\end{aligned}$$



On putting the values of  $I_1$  and  $I_2$  from Eqs. (ii) and (iii) in Eq. (i), we get

$$\begin{aligned}
 I &= \frac{1}{2} [2 \sqrt{x^2 + 5x + 6} + C_1] \\
 &\quad - \frac{1}{2} \left[ \log \left| x + \frac{5}{2} + \sqrt{x^2 + 5x + 6} \right| + C_2 \right] \\
 &= \sqrt{x^2 + 5x + 6} + \frac{C_1}{2} \\
 &\quad - \frac{1}{2} \log \left| x + \frac{5}{2} + \sqrt{x^2 + 5x + 6} \right| - \frac{C_2}{2}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow I &= \sqrt{x^2 + 5x + 6} \\
 &\quad - \frac{1}{2} \log \left| x + \frac{5}{2} + \sqrt{x^2 + 5x + 6} \right| + C \quad \text{(1)}
 \end{aligned}$$

where,  $C = \frac{C_1}{2} - \frac{C_2}{2}$

**38.** Evaluate  $\int (3x - 2) \sqrt{x^2 + x + 1} dx$ .

Foreign 2014

$$\text{Let } I = \int (3x - 2) \sqrt{x^2 + x + 1} dx$$

Here,  $(3x - 2)$  can be written as

$$3x - 2 = A \frac{d}{dx}(x^2 + x + 1) + B = A(2x + 1) + B$$

On equating the coefficients of  $x$  and constant term from both sides, we get  $2A = 3$ ,  $A + B = -2$ , then

$$\Rightarrow A = \frac{3}{2} \text{ and } B = \frac{-7}{2} \quad (1)$$

$$\therefore I = \frac{3}{2} \int (2x + 1) \sqrt{x^2 + x + 1} dx - \frac{7}{2} \int \sqrt{x^2 + x + 1} dx$$

$$\Rightarrow I = I_1 - I_2 \quad \dots(i)$$

$$\text{where, } I_1 = \frac{3}{2} \int (2x + 1) \sqrt{x^2 + x + 1} dx$$

$$\text{Put } x^2 + x + 1 = t$$

$$\Rightarrow 2x + 1 = \frac{dt}{dx}$$

$$\begin{aligned} \therefore I_1 &= \frac{3}{2} \int \sqrt{t} dt = \frac{3}{2} \cdot \frac{2}{3} t^{3/2} + C_1 \\ &= t^{3/2} = (x^2 + x + 1)^{3/2} + C_1 \end{aligned} \quad (1)$$

$$\begin{aligned} \text{and } I_2 &= -\frac{7}{2} \int \sqrt{x^2 + x + 1} dx \\ &= -\frac{7}{2} \int \sqrt{\left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4}} dx \end{aligned}$$

$$\begin{aligned}
&= -\frac{7}{2} \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx \\
&= -\frac{7}{2} \left[ \frac{\left(x + \frac{1}{2}\right)}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2} \left(\frac{\sqrt{3}}{2}\right)^2 \right. \\
&\quad \left. \log \left\{ \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right\} \right] + C_2 \\
&= \frac{-7}{2} \left[ \frac{2x+1}{4} \sqrt{x^2+x+1} + \right. \\
&\quad \left. \frac{3}{8} \log \left\{ \frac{2x+1}{2} + \sqrt{x^2+x+1} \right\} \right] + C_2 \\
&= \frac{-7}{8} (2x+1) \sqrt{x^2+x+1} - \\
&\quad \frac{21}{16} \log \left| \frac{2x+1}{2} + \sqrt{x^2+x+1} \right| + C_2 \\
&\quad \left[ \because \int \sqrt{x^2+a^2} dx = \frac{x}{2} \sqrt{x^2+a^2} \right. \\
&\quad \left. + a^2 \log \left| x + \sqrt{x^2+a^2} \right| + C \right] \quad (1)
\end{aligned}$$

On putting the values of  $I_1$  and  $I_2$  in Eq. (i), we get

$$\begin{aligned}
I &= (x^2+x+1)^{3/2} - \frac{7}{8} (2x+1) \sqrt{x^2+x+1} \\
&\quad - \frac{21}{16} \log \left| \frac{(2x+1)}{2} + \sqrt{x^2+x+1} \right| + C \\
&\text{where, } C = C_1 + C_2 \quad (1)
\end{aligned}$$

**39.** Find  $\int (x+3) \sqrt{3-4x-x^2} dx$ .

Delhi 2014 C

$$\text{Let } I = \int (x+3)\sqrt{3-4x-x^2} dx$$

Here,  $(x+3)$  can be written as

$$(x+3) = A \cdot \frac{d}{dx}(3-4x-x^2) + B$$

$$\Rightarrow (x+3) = A(-4-2x) + B \quad (1)$$

On comparing the coefficients of  $x$  and constant term, we get

$$-2A = 1 \text{ and } -4A + B = 3$$

$$\Rightarrow A = -\frac{1}{2} \text{ and } -4 \times -\frac{1}{2} + B = 3$$

$$\Rightarrow A = -\frac{1}{2} \text{ and } B = 1$$

$$\therefore (x+3) = -\frac{1}{2}(-4-2x) + 1 \quad (1)$$

$$\text{Then, } I = \int \left\{ -\frac{1}{2}(-4-2x) + 1 \right\} \sqrt{3-4x-x^2} dx$$

$$= -\frac{1}{2} \int (-4-2x)\sqrt{3-4x-x^2} dx + \int \sqrt{3-4x-x^2} dx$$

$$= -\frac{1}{2} \int \sqrt{t} dt + \int \sqrt{7-(4x+x^2+4)} dx \quad (1)$$

$$= \left( -\frac{1}{2} \times t^{3/2} \times \frac{2}{3} \right) + \int \sqrt{(\sqrt{7})^2 - (x+2)^2} dx$$

[put  $3-4x-x^2 = t$ ]

$$= -\frac{1}{3}(3-4x-x^2)^{3/2} + \frac{1}{2}(x+2)\sqrt{3-4x-x^2} + \frac{7}{2} \sin^{-1} \left( \frac{x+2}{\sqrt{7}} \right) + C$$

$$\left[ \because \int \sqrt{a^2-x^2} dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \cdot \sin^{-1} \frac{x}{a} \right] \quad (1)$$

**40.** Find  $\int \frac{5x-2}{1+2x+3x^2} dx$ . **Delhi 2014C; Delhi 2013**

Let  $I = \int \frac{5x-2}{1+2x+3x^2} dx \quad \dots(i)$

Here,  $(5x - 2)$  can be written as

$$5x - 2 = A \frac{d}{dx} (1 + 2x + 3x^2) + B$$

$$[\because \text{numerator} = A \cdot \frac{d}{dx} (\text{denominator}) + B]$$

$$\Rightarrow 5x - 2 = A(2 + 6x) + B$$

On comparing the coefficients of  $x$  and constant terms, we get

$$5 = 6A \Rightarrow A = \frac{5}{6}$$

and  $-2 = 2A + B \Rightarrow B = -2A - 2$   
 $= -\frac{5}{3} - 2 = -\frac{11}{3} \quad \left[ \because A = \frac{5}{6} \right]$

Then, from Eq. (i), we get

$$I = \int \frac{\frac{5}{6}(2+6x)}{1+2x+3x^2} dx - \int \frac{\left(\frac{11}{3}\right)}{1+2x+3x^2} dx \quad (1)$$

$$\Rightarrow I = I_1 - I_2 \quad \dots(ii)$$

where,  $I_1 = \frac{5}{6} \int \frac{2+6x}{1+2x+3x^2} dx$

Put  $1+2x+3x^2 = t \Rightarrow (2+6x)dx = dt$

$$\begin{aligned} \therefore I_1 &= \frac{5}{6} \int \frac{dt}{t} = \frac{5}{6} \ln t + C_1 \\ &= \frac{5}{6} \ln |1+2x+3x^2| + C_1 \end{aligned} \quad (1)$$

and  $I_2 = \frac{11}{3} \int \frac{dx}{3x^2+2x+1}$

$$= \frac{11}{9} \int \frac{dx}{\left(x^2 + \frac{2}{3}x + \frac{1}{3} + \frac{1}{9} - \frac{1}{9}\right)}$$

[making a perfect square in denominator]

$$= \frac{11}{9} \int \frac{dx}{\left(x + \frac{1}{3}\right)^2 + \frac{2}{9}}$$

$$= \frac{11}{9} \cdot \frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1} \left| \frac{x + \frac{1}{3}}{\frac{\sqrt{2}}{3}} \right| + C_2$$

$$\left[ \because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \right]$$

$$= \frac{11}{3\sqrt{2}} \tan^{-1} \left| \left( \frac{3x+1}{\sqrt{2}} \right) \right| + C_2 \quad (1)$$

On putting the values of  $I_1$  and  $I_2$  in Eq. (ii), we get


$$I = \frac{5}{6} \ln |1+2x+3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1} \left( \frac{3x+1}{\sqrt{2}} \right) + C$$

where  $C = C_1 + C_2$  (1)



**41.** Find  $\int \frac{x^2}{x^4 + 3x^2 + 2} dx$ .

All India 2014C

 Firstly, put  $x^2 = t$  and use partial fraction to write integral and in simplest form, then integrate by using suitable formula.

Let  $I = \int \frac{x^2}{x^4 + 3x^2 + 2} dx$

Put  $x^2 = t$

$\Rightarrow 2x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2x}$  (1)

$\therefore I = \frac{1}{2} \int \frac{t dt}{t^2 + 3t + 2}$   
 $= \frac{1}{2} \int \frac{t}{(t + 2)(t + 1)} dt$

$= \frac{1}{2} \left[ \int \frac{2}{t + 2} dt - \int \frac{-1}{t + 1} dt \right]$  (1½)  
 [by partial fraction]

$= \frac{1}{2} [2 \log|t + 2| - \log|t + 1|] + C$

$= \log \left| \frac{t + 2}{\sqrt{t + 1}} \right| + C = \log \left| \frac{x^2 + 2}{\sqrt{x^2 + 1}} \right| + C$  (1½)

**42.** Evaluate  $\int \frac{x \cos^{-1} x}{\sqrt{1 - x^2}} dx$ .

All India 2014C; Foreign 2014

Let  $I = \int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$

Put  $\cos^{-1} x = t$

On differentiating both sides w.r.t.  $x$ , we get

$$\frac{-1}{\sqrt{1-x^2}} dx = dt$$

$$\therefore I = -\int t \cos t dt \quad (1)$$

Applying integration by parts, we get

$$\begin{aligned} I &= -\left[ t \int \cos t dt - \int \left( \frac{d}{dt}(t) \int \cos t dt \right) dt \right] \\ &= -[t \sin t - \int \sin t dt] = -t \sin t - \cos t + C \end{aligned} \quad (1\frac{1}{2})$$

Substituting the value of  $t$ , we get

$$\begin{aligned} I &= -\cos^{-1} x \sqrt{1-\cos^2(\cos^{-1} x)} \\ &\quad - \cos(\cos^{-1} x) + C \\ &\quad \left[ \begin{aligned} \because \sin t &= \sqrt{1-\cos^2 t} \\ &= \sqrt{1-\cos^2(\cos^{-1} x)} \end{aligned} \right] \\ \Rightarrow I &= -\left( \sqrt{1-x^2} \right) \cos^{-1} x + C \end{aligned} \quad (1\frac{1}{2})$$

**43.** Evaluate  $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$ .

Delhi 2014C

💡 Firstly, use  $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$  to write numerator of integrand in simplest form and then integrate by using suitable method.

$$\begin{aligned}
 \text{Let } I &= \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx \\
 \Rightarrow I &= \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cos^2 x} dx \\
 &= \int \left[ \frac{(\sin^2 x + \cos^2 x)^3}{\sin^2 x \cos^2 x} - \frac{3 \sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} \right] dx \quad (1) \\
 &\quad [\because a^3 + b^3 = (a + b)^3 - 3ab(a + b)] \\
 &= \int \frac{(1)^3 - 3 \sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx \\
 &\quad [\because \sin^2 x + \cos^2 x = 1] \quad (1\frac{1}{2}) \\
 &= \int \frac{1}{\sin^2 x \cos^2 x} dx - 3 \int \frac{\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx \\
 &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx - 3 \int 1 dx \\
 &= \int \left[ \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right] dx - 3 \int 1 dx \\
 &= \int (\sec^2 x + \operatorname{cosec}^2 x) dx - 3 \int 1 dx \\
 &= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx - 3 \int 1 dx \\
 &= \tan x - \cot x - 3x + C \quad (1\frac{1}{2})
 \end{aligned}$$

44. Evaluate  $\int \frac{(3 \sin x - 2) \cos x}{5 - \cos^2 x - 4 \sin x} dx$ .

Delhi 2013C

$$\begin{aligned} \text{Let } I &= \int \frac{(3\sin x - 2)\cos x}{5 - \cos^2 x - 4\sin x} dx \\ &= \int \frac{(3\sin x - 2)\cos x}{5 - (1 - \sin^2 x) - 4\sin x} dx \end{aligned}$$

$$\text{Put } \sin x = t \Rightarrow \cos x dx = dt$$

$$\therefore I = \int \frac{3t - 2}{5 - (1 - t^2) - 4t} dt \quad (1)$$

$$= \int \frac{3t - 2}{4 + t^2 - 4t} dt = \int \frac{3t - 2}{(t - 2)^2} dt$$

$$= \int \frac{3t - 6 + 4}{(t - 2)^2} dt = \int \frac{3(t - 2) + 4}{(t - 2)^2} dt \quad (1)$$

$$= \int \frac{3}{(t - 2)} dt + \int \frac{4}{(t - 2)^2} dt$$

$$= 3 \log |t - 2| + \frac{4(t - 2)^{-2+1}}{-2 + 1} + C \quad (1)$$

$$\left[ \because \int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1 \right]$$

$$= 3 \log |t - 2| - \frac{4}{(t - 2)} + C$$

$$= 3 \log |\sin x - 2| - \frac{4}{(\sin x - 2)} + C$$

[put  $t = \sin x$ ] (1)

**45.** Evaluate  $\int e^{2x} \left( \frac{1 - \sin 2x}{1 - \cos 2x} \right) dx$ .

Delhi 2013C



Firstly, use trigonometric formulae  $\sin 2\theta = 2\sin\theta\cos\theta$  and  $\cos 2\theta = 1 - 2\sin^2\theta$  to write integral and in simple form, then apply integration by parts to integrate.

$$\begin{aligned}
 \text{Let } I &= \int e^{2x} \left( \frac{1 - \sin 2x}{1 - \cos 2x} \right) dx \\
 &= \int e^{2x} \left( \frac{1 - 2 \sin x \cos x}{2 \sin^2 x} \right) dx \quad (1) \\
 &\quad \left[ \begin{array}{l} \because 1 - \cos 2x = 2 \sin^2 x \\ \text{and } \sin 2x = 2 \sin x \cos x \end{array} \right] \\
 &= \frac{1}{2} \int e^{2x} (\operatorname{cosec}^2 x - 2 \cot x) dx \quad (1\frac{1}{2}) \\
 &= \frac{1}{2} \int e^{2x} \operatorname{cosec}^2 x dx - \int e^{2x} \cot x dx \\
 &= \frac{1}{2} [-e^{2x} \cot x + \int 2e^{2x} \cot x dx] + C \\
 &\quad - \int e^{2x} \cot x dx \\
 &= -\frac{e^{2x}}{2} \cot x + \int e^{2x} \cot x dx \\
 &\quad - \int e^{2x} \cot x dx + C \\
 &= -\frac{e^{2x}}{2} \cot x + C \quad (1\frac{1}{2})
 \end{aligned}$$

**46.** Evaluate  $\int \frac{3x+1}{(x+1)^2(x+3)} dx$ .

Delhi 2013C

$$\text{Let } I = \int \frac{3x+1}{(x+1)^2(x+3)} dx$$

$$\begin{aligned} \text{Again, let } \frac{3x+1}{(x+1)^2(x+3)} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3} \quad \dots(i) \end{aligned}$$

$$\Rightarrow 3x+1 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$$

$$\Rightarrow 3x+1 = A(x^2+4x+3) + B(x+3) + C(x^2+1+2x)$$

$$\Rightarrow 3x+1 = (A+C)x^2 + (4A+B+2C)x + 3A+3B+C \quad \mathbf{(1)}$$

On comparing like powers of  $x$  from both sides, we get

$$A+C=0$$

$$4A+B+2C=3$$

$$\text{and } 3A+3B+C=1$$

On solving, we get

$$A=2, B=-1 \text{ and } C=-2 \quad \mathbf{(1)}$$

$\therefore$  Eq. (i) becomes

$$\frac{3x+1}{(x+1)^2(x+3)} = \frac{2}{x+1} - \frac{1}{(x+1)^2} - \frac{2}{x+3} \quad \mathbf{(1)}$$

On integrating both sides, we get

$$\begin{aligned}
 I &= \int \frac{3x+1}{(x+1)^2(x+3)} dx \\
 &= \int \frac{2}{(x+1)} dx - \int \frac{1}{(x+1)^2} dx - \int \frac{2}{(x+3)} dx \\
 &= 2 \log|x+1| - \frac{(x+1)^{-2+1}}{(-2+1)} - 2 \log|x+3| + C \\
 &= 2 \log \left| \frac{x+1}{x+3} \right| + \frac{1}{(x+1)} + C \\
 &\quad \left[ \because \log m - \log n = \log \frac{m}{n} \right] \text{ (1)}
 \end{aligned}$$

**47.** Evaluate  $\int \frac{\sin(x-a)}{\sin(x+a)} dx$ .

Delhi 2013

$$\text{Let } I = \int \frac{\sin(x-a)}{\sin(x+a)} dx$$

Put  $x+a=t$

$$\Rightarrow x = t - a \Rightarrow dx = dt$$

$$\therefore I = \int \frac{\sin(t-2a)}{\sin t} dt \quad \text{(1)}$$

$$\Rightarrow I = \int \frac{\sin t \cos 2a - \cos t \sin 2a}{\sin t} dt \quad \text{(1)}$$

$$[\because \sin(x-y) = \sin x \cos y - \cos x \sin y]$$

$$\Rightarrow I = \int \frac{\sin t \cos 2a}{\sin t} dt - \int \frac{\cos t \sin 2a}{\sin t} dt$$

$$= \cos 2a \int dt - \sin 2a \int \cot t dt \quad \text{(1)}$$

$$= t \cos 2a - \sin 2a \ln|\sin t| + C$$

$$= (x+a) \cos 2a - \sin 2a \ln |\sin(x+a)| + C \quad \text{(1)}$$

48. Evaluate  $\int \frac{x^2}{(x^2+4)(x^2+9)} dx$ .

Delhi 2013

$$\begin{aligned}\text{Let } I &= \int \frac{x^2}{(x^2+4)(x^2+9)} dx \\ &= \frac{1}{2} \int \frac{2x^2+4+9-4-9}{(x^2+4)(x^2+9)} dx \\ &= \frac{1}{2} \int \frac{x^2+4}{(x^2+4)(x^2+9)} dx + \frac{1}{2} \int \frac{x^2+9}{(x^2+4)(x^2+9)} dx \\ &\quad - \frac{1}{2} \int \frac{13}{(x^2+4)(x^2+9)} dx \quad (1) \\ &= \frac{1}{2} \int \frac{dx}{x^2+9} + \frac{1}{2} \int \frac{dx}{x^2+4} \\ &\quad - \frac{1}{2} \int \frac{13}{(x^2+4)(x^2+9)} dx\end{aligned}$$



$$= \frac{1}{2} \cdot \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2} \cdot \frac{13}{5} \int \left( \frac{1}{(x^2+4)} - \frac{1}{(x^2+9)} \right) dx \quad (1)$$

$$\left[ \because \int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \text{ and apply partial fractions in third term} \right]$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{4} \tan^{-1}\left(\frac{x}{2}\right) - \frac{13}{10} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{13}{10} \cdot \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{4} \tan^{-1}\left(\frac{x}{2}\right) - \frac{13}{20} \tan^{-1}\left(\frac{x}{2}\right) + \frac{13}{30} \tan^{-1}\left(\frac{x}{3}\right) + C \quad (1)$$

$$= \tan^{-1}\left(\frac{x}{3}\right) \left( \frac{1}{6} + \frac{13}{30} \right) + \tan^{-1}\left(\frac{x}{2}\right) \left( \frac{1}{4} - \frac{13}{20} \right) + C$$

$$= \tan^{-1}\left(\frac{x}{3}\right) \left( \frac{5+13}{30} \right) + \tan^{-1}\left(\frac{x}{2}\right) \left( \frac{5-13}{20} \right) + C$$

$$= \frac{18}{30} \tan^{-1}\left(\frac{x}{3}\right) - \frac{8}{20} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \frac{3}{5} \tan^{-1}\left(\frac{x}{3}\right) - \frac{2}{5} \tan^{-1}\left(\frac{x}{2}\right) + C \quad (1)$$

**49.** Evaluate  $\int \frac{2x^2+1}{x^2(x^2+4)} dx$ .

Delhi 2013

$$\text{Let } I = \int \frac{2x^2 + 1}{x^2(x^2 + 4)} dx$$

Put  $x^2 = t$  and then using partial fraction, we get

$$\frac{2t+1}{t(t+4)} = \frac{A}{t} + \frac{B}{t+4}$$

$$\Rightarrow 2t+1 = A(t+4) + Bt \quad (1/2)$$

On comparing the coefficients of  $t$  and constant terms, we get

$$2 = A + B \quad \text{and} \quad 1 = 4A \Rightarrow A = \frac{1}{4}$$

$$\therefore B = 2 - A = 2 - \frac{1}{4} = \frac{7}{4} \quad (1)$$

$$\text{Then, } \frac{2x^2 + 1}{x^2(x^2 + 4)} = \frac{1}{4x^2} + \frac{7}{4(x^2 + 4)}$$

$$\therefore I = \frac{1}{4} \int \frac{dx}{x^2} + \frac{7}{4} \int \frac{dx}{x^2 + 4} \quad (1)$$

$$= -\frac{1}{4x} + \frac{7}{4} \times \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$\left[ \because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \right]$$

$$= -\frac{1}{4x} + \frac{7}{8} \tan^{-1}\left(\frac{x}{2}\right) + C \quad (1\frac{1}{2})$$

**50.** Evaluate  $\int \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} dx$ .

Delhi 2013

$$\text{Let } I = \int \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} dx$$

$$\text{Again, let } \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} = \frac{A}{x^2 + 4} + \frac{B}{x^2 + 25}$$

[by partial fraction] (1)

$$\text{At } x = 0, \quad \frac{A}{4} + \frac{B}{25} = \frac{1}{100}$$

$$\Rightarrow 25A + 4B = 1 \quad \dots(i) \quad (1)$$

$$\text{At } x = 1, \quad \frac{2}{5 \times 26} = \frac{A}{5} + \frac{B}{26}$$

$$\Rightarrow \frac{A}{5} + \frac{B}{26} = \frac{1}{65}$$

$$\Rightarrow 13A + \frac{5}{2}B = 1$$

$$\Rightarrow 26A + 5B = 2 \quad \dots(ii) \quad (1)$$

On solving Eqs. (i) and (ii), we get

$$A = -\frac{1}{7}, \quad B = \frac{8}{7}$$

$$\therefore I = -\frac{1}{7} \int \frac{dx}{x^2 + 4} + \frac{8}{7} \int \frac{dx}{x^2 + 25}$$

$$= -\frac{1}{7} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{7} \cdot \frac{1}{5} \tan^{-1}\left(\frac{x}{5}\right) + C$$

$$\left[ \because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \right] (1)$$

$$= -\frac{1}{14} \tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{35} \tan^{-1}\left(\frac{x}{5}\right) + C$$

**51.** Evaluate  $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$ .

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$$\begin{aligned}
 \text{Let } I &= \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx \\
 &= \int \frac{(2 \cos^2 x - 1) - (2 \cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx \\
 &\quad [\because \cos 2\theta = 2 \cos^2 \theta - 1] \quad (1) \\
 &= \int \frac{2(\cos^2 x - \cos^2 \alpha)}{(\cos x - \cos \alpha)} dx \\
 &= \int \frac{2(\cos x - \cos \alpha)(\cos x + \cos \alpha)}{(\cos x - \cos \alpha)} dx \\
 &\quad [\because a^2 - b^2 = (a + b)(a - b)] \quad (1) \\
 &= \int 2(\cos x + \cos \alpha) dx \\
 &= 2 \left[ \int \cos x dx + \cos \alpha \int dx \right] \\
 \Rightarrow I &= 2(\sin x + x \cos \alpha) + C \quad (2)
 \end{aligned}$$

52. Evaluate  $\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$ .

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Let  $I = \int \frac{x+2}{\sqrt{x^2+2x+3}} dx$

$\Rightarrow I = \int \frac{x+1}{\sqrt{x^2+2x+3}} dx + \int \frac{dx}{\sqrt{x^2+2x+3}}$

$\Rightarrow I = I_1 + I_2 \quad \dots(i) \quad (1)$

where,  $I_1 = \int \frac{(x+1)}{\sqrt{x^2+2x+3}} dx$

$= \int \frac{t}{t} dt = \int dt = t + C_1$

$\left[ \begin{array}{l} \text{put } t^2 = x^2 + 2x + 3 \Rightarrow 2t dt = (2x+2)dx \\ \Rightarrow t dt = (x+1)dx \end{array} \right]$

$= \sqrt{x^2+2x+3} + C_1 \quad (1)$

and  $I_2 = \int \frac{dx}{\sqrt{x^2+2x+3}} = \int \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{2})^2}}$

$= \log \{(x+1) + \sqrt{x^2+2x+3}\} + C_2$

$\left[ \because \int \frac{dx}{\sqrt{x^2+a^2}} = \log(x + \sqrt{x^2+a^2}) \right] (1)$

On putting the values of  $I_1$  and  $I_2$  in Eq. (i), we get

$I = \sqrt{x^2+2x+3} + C_1$   
 $+ \log \{(x+1) + \sqrt{x^2+2x+3}\} + C_2$

$= \sqrt{x^2+2x+3}$

$+ \log \left\{ (x+1) + \sqrt{x^2+2x+3} \right\} + C \quad (1)$

where,  $C = C_1 + C_2$

**53.** Evaluate  $\int \frac{dx}{x(x^5+3)}$ .

HOTS; All India 2013

Let  $I = \int \frac{dx}{x(x^5 + 3)} = \int \frac{x^4}{x^5(x^5 + 3)} dx$

[∴ multiplying numerator and denominator by  $x^4$ ]

Put  $t = x^5 \Rightarrow dt = 5x^4 dx$

∴  $I = \int \frac{dt}{5t(t+3)} \quad (1)$

$$= \frac{1}{5} \int \frac{1}{3} \left[ \frac{1}{t} - \frac{1}{t+3} \right] dt$$

$$= \frac{1}{15} [\log|t| - \log|t+3|] + C \quad (1)$$

$$= \frac{1}{15} \log \left| \frac{t}{t+3} \right| + C \quad (1)$$

54. Evaluate  $\int \frac{dx}{x(x^3 + 1)}$ .

15  $\int \frac{dx}{x^2 + 3}$

All India 2013

Let  $I = \int \frac{dx}{x(x^3+1)}$

$$I = \int \frac{dx}{x(x+1)(x^2-x+1)}$$

$$[\because a^3 + b^3 = (a+b)(a^2 + b^2 - ab)]$$

Again, let

$$\frac{1}{x(x+1)(x^2-x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1}$$

$$1 = (x+1)(x^2-x+1)A + Bx(x^2-x+1)$$

$$+ x(x+1)(Cx+D)$$

$$\Rightarrow 1 = (x^3 - x^2 + x + x^2 - x + 1)A + B(x^3 - x^2 + x)$$

$$+ (x^2 + x)(Cx + D)$$

$$\Rightarrow 1 = A(x^3 + 1) + B(x^3 - x^2 + x)$$

$$+ (Cx^3 + Dx^2 + Cx^2 + Dx)$$

$$\Rightarrow 1 = (A + B + C)x^3 + (-B + D + C)x^2$$

$$+ (B + D)x + A \quad \mathbf{(1)}$$

On comparing the coefficients of different powers of  $x$  from both sides, we get

$$A + B + C = 0 \quad \dots(i)$$

$$-B + D + C = 0 \quad \dots(ii)$$

$$B + D = 0 \quad \dots(iii)$$

and  $A = 1 \quad \dots(iv)$

From Eqs. (ii) and (iii), we get

$$C - 2B = 0 \quad \dots(v)$$

From Eqs. (i) and (iv), we get

$$B + C = -1 \quad \dots(vi)$$

From Eqs. (v) and (vi), we get

$$B = -\frac{1}{3}, D = \frac{1}{3} \text{ and } C = -\frac{2}{3} \quad (1)$$

$$\therefore I = \int \frac{dx}{x(x^3+1)} = \int \left[ \frac{1}{x} - \frac{1}{3(x+1)} + \frac{-\frac{2x}{3} + \frac{1}{3}}{x^2-x+1} \right] dx$$

$$\Rightarrow I = \int \frac{dx}{x} - \frac{1}{3} \int \frac{dx}{x+1} + \frac{1}{3} \int \frac{1-2x}{x^2-x+1} dx$$

$$\Rightarrow I = \log|x| - \frac{1}{3} \log|x+1| - \frac{1}{3} \int \frac{2x-1}{x^2-x+1} dx \quad (1)$$

Put  $t = x^2 - x + 1$

$$\Rightarrow dt = (2x-1)dx$$

$$\therefore I = \log|x| - \frac{1}{3} \log|x+1| - \frac{1}{3} \int \frac{dt}{t}$$

$$= \log|x| - \frac{1}{3} \log|x+1| - \frac{1}{3} \log|t| + C$$

$$= \log|x| - \frac{1}{3} \log|x+1| - \frac{1}{3} \log|x^2-x+1| + C$$

$$[\text{put } t = x^2 - x + 1]$$

$$= \log|x| - \frac{1}{3} \log|(x+1)(x^2-x+1)| + C$$

$$[\because \log m + \log n = \log mn]$$

$$= \log|x| - \frac{1}{3} \log|x^3+1| + C$$

$$= \log|x| - \log|x^3+1|^{1/3} + C = \log \frac{|x|}{|x^3+1|^{1/3}} + C$$

$$\left[ \because \log m - \log n = \log \frac{m}{n} \right] (1)$$

55. Evaluate  $\int \frac{dx}{x(x^3+8)}$ .

All India 2013



$$\begin{aligned} \text{Let } I &= \int \frac{dx}{x(x^3+8)} = \int \frac{dx}{x(x^3+2^3)} \\ &= \int \frac{dx}{x(x+2)(x^2-2x+4)} \quad \dots(i) \end{aligned}$$

$$[\because x^3 + a^3 = (x+a)(x^2 + a^2 - ax)]$$

Now, for partial fraction method, write

$$\begin{aligned} \frac{1}{x(x+2)(x^2-2x+4)} &= \frac{A}{x} + \frac{B}{x+2} + \frac{Cx+D}{x^2-2x+4} \\ \Rightarrow 1 &= A(x+2)(x^2-2x+4) + Bx(x^2-2x+4) \\ &\quad + (Cx+D)(x^2+2x) \end{aligned}$$

$$\begin{aligned} \Rightarrow 1 &= A(x^3 - 2x^2 + 4x + 2x^2 - 4x + 8) \\ &\quad + B(x^3 - 2x^2 + 4x) + (Cx^3 + 2Cx^2 + Dx^2 + 2Dx) \end{aligned}$$

$$\begin{aligned} \Rightarrow 1 &= A(x^3 + 8) + B(x^3 - 2x^2 + 4x) \\ &\quad + (Cx^3 + 2Cx^2 + Dx^2 + 2Dx) \quad (1) \end{aligned}$$

$$\begin{aligned} \Rightarrow 1 &= (A+B+C)x^3 + (-2B+2C+D)x^2 \\ &\quad + (4B+2D)x + 8A \end{aligned}$$

On comparing the coefficients of different powers of  $x$  from both sides, we get

$$A+B+C=0 \quad \dots(ii)$$

$$-2B+2C+D=0 \quad \dots(iii)$$

$$4B+2D=0 \quad \dots(iv)$$

$$\text{and} \quad 8A=1$$

$$\therefore A = \frac{1}{8}$$

From Eqs. (iii) and (iv), we get

$$-2B+2C-2B=0 \Rightarrow -4B+2C=0$$

$$\therefore C=2B \quad \dots(v)$$

On putting the values of  $C$  and  $A$  in Eq. (ii), we get

$$\frac{1}{8} + B + 2B = 0$$

$$\Rightarrow B = -\frac{1}{24} \text{ and } C = -\frac{1}{12}$$

and  $D = \frac{1}{12}$  (1)

Now, substituting the values of A, B, C and D in the given integral, we get

$$\frac{1}{x(x+2)(x^2-2x+4)}$$

$$= \frac{1}{8x} - \frac{1}{24(x+2)} + \frac{-\frac{1}{12}x + \frac{1}{12}}{x^2-2x+4}$$

$$\therefore I = \int \left[ \frac{1}{8x} - \frac{1}{24(x+2)} + \frac{\left(\frac{-x}{12} + \frac{1}{12}\right)}{x^2-2x+4} \right] dx$$

$$= \frac{1}{8} \int \frac{dx}{x} - \frac{1}{24} \int \frac{dx}{x+2}$$

$$- \frac{1}{12} \int \frac{x-1}{x^2-2x+4} dx \text{ (1)}$$

$$= \frac{1}{8} \log|x| - \frac{1}{24} \log|x+2|$$

$$- \frac{1}{24} \int \frac{2x-2}{x^2-2x+4} dx$$

$$\begin{aligned}
&= \frac{1}{8} \log|x| - \frac{1}{24} \log|x+2| \\
&\quad - \frac{1}{24} \log|x^2 - 2x + 4| + C \\
&\quad [\text{let } x^2 - 2x + 4 = t \Rightarrow (2x - 2)dx = dt] \\
&= \frac{1}{8} \log|x| - \frac{1}{24} \log|(x+2)(x^2 - 2x + 4)| + C \\
&\quad [:\log m + \log n = \log mn] \\
&= \frac{1}{8} \log|x| - \frac{1}{24} \log|(x^3 + 8)| + C \\
&\quad [:(a + b)(a^2 - ab + b^2) = a^3 + b^3] \\
&= \frac{1}{8} \left\{ \log|x| - \frac{1}{3} \log|x^3 + 8| \right\} + C \\
&= \frac{1}{8} \{ \log|x| - \log|x^3 + 8|^{1/3} \} + C \\
&= \frac{1}{8} \log \left| \frac{x}{(x^3 + 8)^{1/3}} \right| + C \qquad (1)
\end{aligned}$$

**56.** Evaluate  $\int \sin x \cdot \sin 2x \cdot \sin 3x dx$  **HOTS; Delhi 2012**

? It is a product of three trigonometric functions. So, firstly we take two functions at a time and use the relation  $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$  and then integrate it.

$$\text{Let } I = \int \sin x \sin 2x \sin 3x \, dx$$

$$= \frac{1}{2} \int \sin x (2 \sin 2x \sin 3x) \, dx$$

[multiplying numerator and denominator by 2]

$$= \frac{1}{2} \int \sin x [\cos(2x - 3x) - \cos(2x + 3x)] \, dx$$

$$[\because 2 \sin A \sin B = \cos(A - B) - \cos(A + B)] \quad (1)$$

$$= \frac{1}{2} \int \sin x [\cos(-x) - \cos 5x] \, dx$$

$$= \frac{1}{2} \int \sin x (\cos x - \cos 5x) \, dx \quad [\because \cos(-x) = \cos x]$$

$$= \frac{1}{2} \int \sin x \cos x \, dx - \frac{1}{2} \int \sin x \cos 5x \, dx \quad (1)$$

$$= \frac{1}{4} \int 2 \sin x \cos x \, dx - \frac{1}{4} \int (2 \sin x \cos 5x) \, dx$$

[multiplying numerator and denominator by 2]

$$= \frac{1}{4} \int \sin 2x \, dx - \frac{1}{4} \int \{\sin(x + 5x)$$

$$\left[ \begin{array}{l} \because 2 \sin x \cos x = \sin 2x \text{ and} \\ 2 \sin A \cos B = \sin(A + B) + \sin(A - B) \end{array} \right]$$

$$= \frac{1}{4} \int \sin 2x \, dx - \frac{1}{4} \int [\sin 6x + \sin(-4x)] \, dx \quad (1)$$

$$= \frac{1}{4} \int \sin 2x \, dx - \frac{1}{4} \int (\sin 6x - \sin 4x) \, dx$$

$$[\because \sin(-\theta) = -\sin \theta]$$


$$= \frac{-1}{4} \cdot \frac{\cos 2x}{2} - \frac{1}{4} \left[ \frac{-\cos 6x}{6} + \frac{\cos 4x}{4} \right] + C$$

$$\left[ \because \int \sin ax \, dx = \frac{-\cos ax}{a} \right]$$

$$= \frac{-\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C \quad (1)$$

57. Evaluate  $\int \frac{2}{(1-x)(1+x^2)} dx$ .

Delhi 2012

 Here, denominator is a product of two algebraic functions. So, firstly we use partial fraction method and then integrate it.

Let 
$$I = \int \frac{2}{(1-x)(1+x^2)} dx$$

Again, let 
$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2} \dots(i)$$

(1)

$$\Rightarrow \frac{2}{(1-x)(1+x^2)} = \frac{A(1+x^2) + (Bx+C)(1-x)}{(1-x)(1+x^2)}$$

$$\Rightarrow 2 = A(1+x^2) + (Bx+C)(1-x)$$

$$\Rightarrow 2 = A + Ax^2 + Bx + C - Bx^2 - Cx$$

$$\Rightarrow 2 = (A-B)x^2 + (B-C)x + (A+C)$$

(1)

On comparing coefficients of  $x^2$ ,  $x$  and constant terms from both sides, we get

$$A - B = 0 \dots(ii)$$

$$B - C = 0 \dots(iii)$$

and  $A + C = 2 \dots(iv)$

On adding Eqs. (ii) and (iii), we get

$$A - C = 0 \dots(v)$$

On adding Eqs. (iv) and (v), we get

$$2A = 2 \Rightarrow A = 1$$

On putting  $A = 1$  in Eq. (ii), we get  $B = 1$ .

On putting  $B = 1$  in Eq. (iii), we get  $C = 1$ .

Hence,  $A = 1, B = 1$  and  $C = 1$  (1)

Now, Eq. (i) becomes

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

On integrating both sides w.r.t.  $x$ , we get

$$\int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x+1}{1+x^2} dx$$

$$= -\log|1-x| + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\log|1-x| + \frac{1}{2} \log|1+x^2| + \tan^{-1} x + C$$

$$\left[ \begin{array}{l} \text{put } 1+x^2 = t \\ \Rightarrow 2x dx = dt \Rightarrow x dx = dt/2 \\ \therefore \int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log|t| = \frac{1}{2} \log|1+x^2| \end{array} \right]$$

(1)

58. Evaluate  $\int \left( \frac{1+\sin x}{1+\cos x} \right) e^x dx$ .

All India 2012C

$$\begin{aligned} \text{Let } I &= \int \left( \frac{1 + \sin x}{1 + \cos x} \right) e^x dx \\ &= \int \frac{1 + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \cdot e^x dx \end{aligned} \quad (1)$$

$$\left[ \begin{array}{l} \because \sin m = 2 \sin \frac{m}{2} \cos \frac{m}{2} \\ \text{and } 1 + \cos m = 2 \cos^2 \frac{m}{2} \end{array} \right]$$

$$= \int \left( \frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) e^x dx \quad (1)$$

$$= \int e^x \left( \tan \frac{x}{2} + \frac{1}{2} \sec^2 \frac{x}{2} \right) dx \quad (1)$$

On comparing it with

$$\int e^x [f(x) + f'(x)] dx = e^x f(x), \text{ we get}$$

$$f(x) = \tan \frac{x}{2} \Rightarrow f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$$

$$\therefore I = e^x \tan \frac{x}{2} + C \quad (1)$$

**59.** Evaluate  $\int \frac{x^2}{(x \sin x + \cos x)^2} dx$ .

All India 2012C

$$\text{Let } I = \int \frac{x^2}{(x \sin x + \cos x)^2} dx$$

$$\Rightarrow I = \frac{x \cos x}{(x \sin x + \cos x)^2} \cdot x \sec x dx \quad \dots(i)(1)$$

$$\text{Again, let } x \sin x + \cos x = t$$

$$\Rightarrow (x \cos x + \sin x - \sin x) dx = dt$$

$$\Rightarrow x \cos x dx = dt$$

$$\begin{aligned} \therefore I_1 &= \int \frac{x \cos x dx}{(x \sin x + \cos x)^2} \\ &= \int \frac{dt}{t^2} = \frac{-1}{t} = \frac{-1}{x \sin x + \cos x} \end{aligned} \quad (1)$$

Now, integrating Eq. (i) by parts, we get

$$\begin{aligned}
 I &= \int x \sec x \cdot \frac{x \cos x}{(x \sin x + \cos x)^2} dx \\
 &= x \sec x \cdot \frac{(-1)}{x \sin x + \cos x} \\
 &\quad - \int (1 \cdot \sec x + x \sec x \tan x) \cdot \frac{-dx}{x \sin x + \cos x} \quad (1) \\
 &= \frac{-x \sec x}{x \sin x + \cos x} + \int \sec x \left(1 + \frac{x \sin x}{\cos x}\right) \frac{dx}{x \sin x + \cos x} \\
 &= \frac{-x \sec x}{x \sin x + \cos x} + \int \sec^2 x dx \\
 &= \frac{-x \sec x}{x \sin x + \cos x} + \tan x + C \quad (1)
 \end{aligned}$$

**60.** Evaluate  $\int e^{2x} \sin x dx$ .

Foreign 2011



Let  $I = \int e^{2x} \sin x \, dx$

On taking  $\sin x$  as I function and  $e^{2x}$  as II function and integrating by parts, we get

$$I = \sin x \int e^{2x} \, dx - \int \left\{ \frac{d}{dx} (\sin x) \int e^{2x} \, dx \right\} dx \quad (1)$$

$$\Rightarrow I = \frac{\sin x \cdot e^{2x}}{2} - \int \frac{\cos x \cdot e^{2x}}{2} \, dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} I_1 \quad \dots(i) \quad (1)$$

where,  $I_1 = \int e^{2x} \cos x \, dx$

On integrating by parts again by taking  $\cos x$  as I function and  $e^{2x}$  as II function, we get

$$I_1 = \cos x \int e^{2x} \, dx - \int \left\{ \frac{d}{dx} (\cos x) \int e^{2x} \, dx \right\} dx$$

$$\Rightarrow I_1 = \frac{\cos x \cdot e^{2x}}{2} - \int \frac{(-\sin x) \cdot e^{2x}}{2} \, dx$$

$$\Rightarrow I_1 = \frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x \, dx$$

$$\Rightarrow I_1 = \frac{e^{2x} \cos x}{2} + \frac{1}{2} I \quad \dots(ii)$$

$$\left[ \because \int e^{2x} \sin x \, dx = I \right] \quad (1)$$

On putting the value of  $I_1$  from Eq. (ii) in Eq. (i), we get

$$I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[ \frac{e^{2x} \cos x}{2} + \frac{1}{2} I \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I$$

$$\Rightarrow I + \frac{1}{4} I = e^{2x} \left( \frac{\sin x}{2} - \frac{\cos x}{4} \right)$$

$$\Rightarrow \frac{5I}{4} = e^{2x} \left( \frac{2 \sin x - \cos x}{4} \right)$$

$$\Rightarrow I = \frac{4}{5} e^{2x} \left( \frac{2 \sin x - \cos x}{4} \right)$$

$$\Rightarrow I = 1/5 e^{2x} (2 \sin x - \cos x) \quad (1)$$

**61.** Evaluate  $\int \frac{3x + 5}{\sqrt{x^2 - 8x + 7}} dx$ .

Foreign 2011

Let 
$$I = \int \frac{3x + 5}{\sqrt{x^2 - 8x + 7}} dx$$

Here,  $(3x + 5)$  can be written as

$$3x + 5 = A \cdot \frac{d}{dx} (x^2 - 8x + 7) + B$$

$$\Rightarrow 3x + 5 = A(2x - 8) + B \quad \dots(i)$$

$$\Rightarrow 3x + 5 = 2Ax + (B - 8A) \quad (1/2)$$

On comparing the coefficients of  $x$  and constant terms from both sides, we get

$$2A = 3 \quad \dots(ii)$$

$$\text{and } -8A + B = 5 \quad \dots(iii)$$

From Eq. (ii), we get  $A = \frac{3}{2}$

On putting  $A = \frac{3}{2}$  in Eq. (iii), we get

$$-8 \left( \frac{3}{2} \right) + B = 5$$

$$\Rightarrow -12 + B = 5$$

$$\Rightarrow B = 17 \quad (1/2)$$

On putting the values of  $A$  and  $B$  in Eq. (i), we get

$$3x + 5 = \frac{3}{2} (2x - 8) + 17 \quad \dots(iv)$$

Hence, the given integral can be written as

$$I = \int \frac{\frac{3}{2} (2x - 8) + 17}{\sqrt{x^2 - 8x + 7}} dx \quad [\text{using Eq. (iv)}]$$

$$\Rightarrow I = \frac{3}{2} \int \frac{2x - 8}{\sqrt{x^2 - 8x + 7}} dx$$

$$+ 17 \int \frac{dx}{\sqrt{x^2 - 8x + 7}}$$

$$\Rightarrow I = \frac{3}{2} I_1 + 17 I_2 \quad \dots(v) (1)$$

$$I_1 = \int \frac{2x - 8}{\sqrt{x^2 - 8x + 7}} dx$$

where,  $I_1 = \int \frac{dx}{\sqrt{x^2 - 8x + 7}}$

Put  $x^2 - 8x + 7 = t$

$\Rightarrow (2x - 8) dx = dt$

$\therefore I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} dt$

$\Rightarrow I_1 = \frac{t^{1/2}}{1/2}$

$\Rightarrow I_1 = 2t^{1/2} \Rightarrow I_1 = 2\sqrt{t}$  **(1/2)**

$\Rightarrow I_1 = 2\sqrt{x^2 - 8x + 7}$

and  $I_2 = \int \frac{dx}{\sqrt{x^2 - 8x + 7}}$

$\Rightarrow I_2 = \int \frac{dx}{\sqrt{x^2 - 8x + 7 + 16 - 16}}$

$\Rightarrow I_2 = \int \frac{dx}{\sqrt{(x-4)^2 - 9}}$   
 $= \int \frac{dx}{\sqrt{(x-4)^2 - (3)^2}}$

$\therefore I_2 = \log |(x-4) + \sqrt{(x-4)^2 - (3)^2}|$

$\left[ \because \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log |x + \sqrt{x^2 - a^2}| \right]$  **(1)**

Hence, putting the values of  $I_1$  and  $I_2$  in Eq. (v), we get

$I = \frac{3}{2} (2\sqrt{x^2 - 8x + 7})$


$+ 17 \log |(x-4) + \sqrt{(x-4)^2 - 9}| + C$

$\Rightarrow I = 3\sqrt{x^2 - 8x + 7} + 17 \log |(x-4)$

$+ \sqrt{(x-4)^2 - 9}| + C$  **(1/2)**

**62.** Evaluate  $\int \frac{x^2 + 4}{x^4 + 16} dx$ .

All India 2011C

 Firstly, divide numerator and denominator by  $x^2$  and reduce the integrand in standard form.

Let 
$$I = \int \frac{x^2 + 4}{x^4 + 16} dx$$

On dividing numerator and denominator by  $x^2$ , we get

$$I = \int \frac{\left(1 + \frac{4}{x^2}\right)}{\left(x^2 + \frac{16}{x^2}\right)} dx = \int \frac{\left(1 + \frac{4}{x^2}\right)}{\left(x - \frac{4}{x}\right)^2 + 8} dx \quad (1)$$

Put  $x - \frac{4}{x} = t \Rightarrow \left(1 + \frac{4}{x^2}\right) dx = dt$

$$\therefore I = \int \frac{dt}{t^2 + 8} = \int \frac{dt}{t^2 + (2\sqrt{2})^2} \quad (1)$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1} \left( \frac{t}{2\sqrt{2}} \right) + C$$

$$\left[ \because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \right]$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1} \left( \frac{x - \frac{4}{x}}{2\sqrt{2}} \right) + C$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1} \left( \frac{x^2 - 4}{2\sqrt{2}x} \right) + C \quad (2)$$

**63.** Evaluate  $\int \frac{x^2 + 1}{x^4 + 1} dx$ .

Delhi 2011C

$$\text{Let } I = \int \frac{x^2 + 1}{x^4 + 1} dx$$

On dividing numerator and denominator by  $x^2$ , we get

$$\begin{aligned} I &= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2}\right)} dx = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2} + 2 - 2\right)} dx \\ &= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{2})^2} dx \quad (1\frac{1}{2}) \end{aligned}$$

$$\text{Put } x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt \quad (1)$$

$$\therefore I = \int \frac{dt}{t^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{t}{\sqrt{2}} \right) + C$$

$$\left[ \because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \right]$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{x - \frac{1}{x}}{\sqrt{2}} \right) + C \quad (1)$$

$$I = \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{x^2 - 1}{x\sqrt{2}} \right) + C \quad (1/2)$$

**64.** Evaluate  $\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$ .

HOTS; Delhi 2011C

💡 In this type of integral, first we make a term in denominator such that whose differential coefficient present in numerator and then integrate it.

$$\text{Let } I = \int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$$

$$= \int \frac{\sin x - \cos x}{\sqrt{1 + \sin 2x - 1}} dx \quad (1)$$

$$= \int \frac{\sin x - \cos x}{\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x - 1}} dx$$

[ $\because \sin^2 \theta + \cos^2 \theta = 1$ ]

$$= \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 1}} dx \quad (1)$$

Put  $\sin x + \cos x = t$

$\Rightarrow (\cos x - \sin x) dx = dt$

$$\therefore I = \int \frac{-dt}{\sqrt{t^2 - 1}} = -\log(t + \sqrt{t^2 - 1}) + C$$

$$\left[ \because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2}) \right] (1)$$

$\Rightarrow I = -\log |(\sin x + \cos x)$

$+ \sqrt{(\sin x + \cos x)^2 - 1}| + C$

[ $\because t = \sin x + \cos x$ ]

$= -\log |(\sin x + \cos x) + \sqrt{\sin 2x}| + C \quad (1)$

65. Evaluate  $\int \frac{2x}{(x^2 + 1)(x^2 + 3)} dx$ .

Delhi 2011

Let  $I = \int \frac{2x}{(1 + x^2)(x^2 + 3)} dx$

Put  $x^2 = t$

$\Rightarrow 2x dx = dt$  (1)

$\therefore I = \int \frac{dt}{(t + 1)(3 + t)}$

Again, let  $\frac{1}{(t + 1)(3 + t)} = \frac{A}{(1 + t)} + \frac{B}{3 + t}$  ... (i)

$\Rightarrow 1 = A(3 + t) + B(1 + t)$  (1)



On putting  $t = -3$ , we get

$$1 = -2B \Rightarrow B = -\frac{1}{2}$$

Now, on putting  $t = -1$ , we get

$$1 = 2A \Rightarrow A = 1/2$$

On putting  $A = \frac{1}{2}$  and  $B = -\frac{1}{2}$  in Eq. (i), we get

$$\frac{1}{(1+t)(3+t)} = \frac{1/2}{1+t} + \frac{-1/2}{3+t} \quad (1/2)$$

On integrating both sides, we get

$$\begin{aligned} \int \frac{1}{(1+t)(3+t)} dt &= \frac{1}{2} \int \frac{1}{1+t} dt - \frac{1}{2} \int \frac{1}{3+t} dt \\ &= \frac{1}{2} \log|1+t| - \frac{1}{2} \log|3+t| \\ &\quad \left[ \because \int \frac{dx}{x} = \log|x| \right] \\ &= \frac{1}{2} \log|1+x^2| - \frac{1}{2} \log|3+x^2| + C \\ &\quad [\because t = x^2] \end{aligned}$$

$$\begin{aligned} \therefore I &= \frac{1}{2} \log \left| \frac{1+x^2}{3+x^2} \right| + C \\ &\quad \left[ \because \log m - \log n = \log \frac{m}{n} \right] (1\frac{1}{2}) \end{aligned}$$

**66.** Evaluate  $\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx$ .

Delhi 2011; All India 2010

Do same as Que 37.

[Ans.  $5\sqrt{x^2+4x+10}$

$$-7 \log|x+2+\sqrt{x^2+4x+10}| + C]$$

**67.** Evaluate  $\int e^{2x} \left( \frac{1+\sin 2x}{1+\cos 2x} \right) dx$ . All India 2010C

Do same as Que 45. [Ans.  $\frac{1}{2} e^{2x} \tan x + C$ ]

68. Evaluate  $\int \frac{dx}{(x^2 + 1)(x^2 + 2)}$ . Delhi 2010C

Do same as Que 65.

$$\left[ \text{Ans. } \tan^{-1} x - \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{x}{\sqrt{2}} \right) + C \right]$$

69. Evaluate  $\int \left[ \log(\log x) + \frac{1}{(\log x)^2} \right] dx$ .  
HOTS; Delhi 2010C



Use integration by parts i.e.

$$\int u \cdot v dx = \left[ u \int v dx - \int \left\{ \frac{d}{dx} u \int v dx \right\} dx \right] \text{ and also}$$

remember ILATE whenever use it.

$$\begin{aligned} \text{Let } I &= \int \left[ \log(\log x) + \frac{1}{(\log x)^2} \right] dx \\ &= \int \log(\log x) \cdot 1 dx + \int \frac{1}{(\log x)^2} dx \quad (1/2) \end{aligned}$$



Using integration by parts in first integral, we get

$$\begin{aligned}
 I &= \log(\log x) \int 1 dx - \int \left[ \frac{d}{dx} \log(\log x) \int 1 dx \right] dx \\
 &\quad + \int \frac{1}{(\log x)^2} dx + C \quad (1/2) \\
 &= \log(\log x) \cdot x - \int \frac{1}{\log x} \cdot \frac{1}{x} \cdot x dx \\
 &\quad + \int \frac{1}{(\log x)^2} dx + C \\
 &= x \log(\log x) - \int (\log x)^{-1} 1 dx \\
 &\quad + \int \frac{1}{(\log x)^2} dx + C \quad (1)
 \end{aligned}$$

Again, applying integration by parts in the middle integral, we get

$$\begin{aligned}
 I &= x \log(\log x) - [(\log x)^{-1} \int 1 dx \\
 &\quad - \int \left\{ \frac{d}{dx} (\log x)^{-1} \int 1 dx \right\} dx] + \int \frac{1}{(\log x)^2} dx + C \\
 &\quad (1) \\
 &= x \log(\log x) - \left[ \frac{x}{\log x} - \int -(\log x)^{-2} \cdot \frac{1}{x} \cdot x dx \right] \\
 &\quad + \int \frac{1}{(\log x)^2} dx + C \\
 &= x \log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2} dx \\
 &\quad + \int \frac{1}{(\log x)^2} dx + C \\
 &= x \log(\log x) - \frac{x}{\log x} + C \quad (1)
 \end{aligned}$$

**70.** Evaluate  $\int \frac{x+2}{\sqrt{(x-2)(x-3)}} dx$ .

All India 2010

Do same as Que 61.

$$\left[ \text{Ans. } \sqrt{x^2 - 5x + 6} + \frac{9}{2} \log \left| \left( x - \frac{5}{2} \right) + \sqrt{(x-2)(x-3)} \right| + C \right]$$

**71.** Evaluate  $\int \frac{1-x^2}{x(1-2x)} dx$ .

Delhi 2010

Let  $I = \int \frac{1-x^2}{x(1-2x)} dx = \int \frac{1-x^2}{x-2x^2} dx$

Given integral can be rewritten as

$$I = \int \left[ \frac{1}{2} + \frac{1 - \frac{1}{2}x}{x(1-2x)} \right] dx \quad (1)$$

$$\Rightarrow I = \frac{1}{2} \int dx + \int \frac{1 - \frac{1}{2}x}{x(1-2x)} dx \dots(i)$$

$$\text{Let } \frac{\left(1 - \frac{1}{2}x\right)}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x} \dots(ii)$$

$$\Rightarrow 1 - \frac{1}{2}x = A(1-2x) + Bx \dots(iii) \text{ (1)}$$

On putting  $x = 0$  and  $x = \frac{1}{2}$  in Eq. (iii), we get

$$1 - 0 = A(1 - 0) + 0$$

$$\Rightarrow A = 1$$

$$\text{and } 1 - \frac{1}{2}\left(\frac{1}{2}\right) = A\left[1 - 2\left(\frac{1}{2}\right)\right] + B\left(\frac{1}{2}\right)$$

$$\Rightarrow 1 - \frac{1}{4} = A(1 - 1) + \frac{1}{2}B$$

$$\Rightarrow \frac{3}{4} = \frac{1}{2}B$$

$$\Rightarrow B = \frac{3}{2} \text{ (1)}$$

On putting the values of  $A$  and  $B$  in Eq (ii), we get

$$\frac{1 - \frac{1}{2}x}{x(1-2x)} = \frac{1}{x} + \frac{3/2}{1-2x}$$

Then, from Eq. (i), we get

$$I = \frac{1}{2} \int dx + \int \frac{1}{x} dx + \int \frac{3/2}{1-2x} dx$$

$$= \frac{1}{2}x + \log|x| + \frac{3}{2} \frac{\log|1-2x|}{-2} + C$$

$$\left[ \because \int \frac{1}{a-x} dx = -\frac{1}{a} \log|a-x| \right]$$

$$= \frac{1}{2}x + \log|x| - \frac{3}{4} \log|1-2x| + C \text{ (1)}$$

**72.** Evaluate  $\int e^x \left( \frac{\sin 4x - 4}{1 - \cos 4x} \right) dx.$

Delhi 2010

$$\begin{aligned} \text{Let } I &= \int e^x \left( \frac{\sin 4x - 4}{1 - \cos 4x} \right) dx \\ &= \int e^x \left( \frac{2 \sin 2x \cos 2x - 4}{2 \sin^2 2x} \right) dx \quad (1) \end{aligned}$$

$$= \int e^x \left( \frac{2 \sin 2x \cos 2x}{2 \sin^2 2x} - \frac{4}{2 \sin^2 2x} \right) dx \quad (1)$$

$$= \int e^x (\cot 2x - 2 \operatorname{cosec}^2 2x) dx$$

On comparing with

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C, \text{ we get}$$

$$f(x) = \cot 2x$$

$$\Rightarrow f'(x) = -2 \operatorname{cosec}^2 2x$$

$$\therefore I = e^x \cot 2x + C \quad (2)$$

**73.** Evaluate  $\int \frac{(\sin x + \cos x)}{\sqrt{\sin 2x}} dx.$

HOTS; All India 2009C

Do same as Ques 64.

$$[\text{Ans. } \sin^{-1}(\sin x - \cos x) + C]$$

$$\text{use } \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

**74.** Evaluate  $\int \frac{2x + 5}{\sqrt{7 - 6x - x^2}} dx.$

Delhi 2009C

$$\text{Let } I = \int \frac{2x + 5}{\sqrt{7 - 6x - x^2}} dx$$

Given integral can be written as

$$\begin{aligned} I &= \int \frac{-(-2x - 6) - 1}{\sqrt{7 - 6x - x^2}} dx \\ &= - \int \frac{-2x - 6}{\sqrt{7 - 6x - x^2}} dx - \int \frac{dx}{\sqrt{7 - 6x - x^2}} \end{aligned}$$

$$\Rightarrow I = -I_1 - I_2 \quad \dots(i) \quad (1)$$

$$\text{where, } I_1 = \int \frac{-2x - 6}{\sqrt{7 - 6x - x^2}} dx$$

$$\text{Put } 7 - 6x - x^2 = t$$

$$\Rightarrow (-6 - 2x) dx = dt$$

$$\begin{aligned} \therefore I_1 &= \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} dt \\ &= 2\sqrt{t} = 2\sqrt{7 - 6x - x^2} \quad (1) \end{aligned}$$

$$\begin{aligned} \text{And } I_2 &= \int \frac{dx}{\sqrt{7 - 6x - x^2}} \\ &= \int \frac{dx}{\sqrt{-(-7 + 6x + x^2 + 9 - 9)}} \\ &= \int \frac{dx}{\sqrt{-[(x + 3)^2 - 16]}} \\ \Rightarrow I_2 &= \int \frac{dx}{\sqrt{(4)^2 - (x + 3)^2}} \\ &= \sin^{-1} \left( \frac{x + 3}{4} \right) + C \quad (1) \end{aligned}$$


$$\left[ \because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \right]$$

On putting the values of  $I_1$  and  $I_2$  in Eq. (i), we get

$$I = -2\sqrt{7 - 6x - x^2} - \sin^{-1}\left(\frac{x+3}{4}\right) + C \quad (1)$$

**75.** Evaluate  $\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$

All India 2009

 Firstly, make a perfect square in denominator part and then integrate it using suitable formula.

$$\begin{aligned} \text{Let } I &= \int \frac{dx}{\sqrt{5 - 4x - 2x^2}} \\ &= \int \frac{dx}{\sqrt{-2\left(x^2 + 2x - \frac{5}{2}\right)}} \\ &= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left(x^2 + 2x - \frac{5}{2} + 1 - 1\right)}} \\ &= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left[(x^2 + 2x + 1) - \frac{5}{2} - 1\right]}} \quad (1) \\ &= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left[(x+1)^2 - \left(\frac{5}{2} + 1\right)\right]}} \\ &= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left[(x+1)^2 - \frac{7}{2}\right]}} \quad (1) \\ &= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{\sqrt{7}}{\sqrt{2}}\right)^2 - (x+1)^2}} \\ &= \frac{1}{\sqrt{2}} \cdot \sin^{-1}\left[\frac{x+1}{\frac{\sqrt{7}}{\sqrt{2}}}\right] + C \quad (1) \end{aligned}$$



$$\begin{aligned} & \left( \sqrt{2} \right) \\ & \left[ \because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \right] \\ & = \frac{1}{\sqrt{2}} \sin^{-1} \left[ \frac{\sqrt{2}(x+1)}{\sqrt{7}} \right] + C \quad (1) \end{aligned}$$

**76.** Evaluate  $\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} dx$ . **HOTS; Delhi 2009**

Firstly, put  $e^x = t \Rightarrow e^x dx = dt$ , Then, do same as Que 75.

$$\left[ \text{Ans. } \sin^{-1} \left( \frac{e^x + 2}{3} \right) + C \right]$$

**77.** Evaluate  $\int \frac{x+3}{x^2 - 2x - 5} dx$ . **All India 2008C**

Let  $I = \int \frac{x+3}{x^2 - 2x - 5} dx$

Here  $(x+3)$  can be written as

$$x+3 = A + B \frac{d}{dx} (x^2 - 2x - 5)$$

$$\Rightarrow x+3 = A + B(2x-2) \quad (1)$$

On equating the coefficients of like terms from both sides, we get

$$2B = 1 \Rightarrow B = 1/2$$

and  $A - 2B = 3$

$$\Rightarrow A - 2 \times \frac{1}{2} = 3$$

$$\Rightarrow A = 3 + 1 = 4 \quad (1)$$

$$\therefore I = \int \frac{4 + \frac{1}{2}(2x-2)}{x^2 - 2x - 5} dx$$

$$\Rightarrow I = 4 \int \frac{1}{x^2 - 2x - 5} dx$$

$$+ \frac{1}{2} \int \frac{2x-2}{x^2 - 2x - 5} dx \quad (1)$$

$$t = x^2 - 2x - 5$$

On putting  $x^2 - 2x - 5 = t \Rightarrow (2x - 2) dx = dt$   
in second integral, we get

$$\begin{aligned} I &= 4 \int \frac{dx}{(x-1)^2 - 1 - 5} + \frac{1}{2} \int \frac{dt}{t} \\ &= 4 \int \frac{dx}{(x-1)^2 - (\sqrt{6})^2} + \frac{1}{2} \log |t| + C \\ &= 4 \cdot \frac{1}{2\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| \\ &\quad + \frac{1}{2} \log |x^2 - 2x - 5| + C \\ &\quad \left[ \because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \right] \\ &= \frac{2}{\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| \\ &\quad + \frac{1}{2} \log |x^2 - 2x - 5| + C \quad (1) \end{aligned}$$

**78.** Evaluate  $\int x \sin^{-1} x \, dx$ .

All India 2008C



Use integration by parts

$$\int u \cdot v \, dx = \left[ u \int v \, dx - \int \left\{ \frac{d}{dx} u \int v \, dx \right\} dx \right]$$

Let  $I = \int x \sin^{-1} x \, dx$

Taking  $x$  as 1st function and  $\sin^{-1} x$  as 2nd function and using the rule of integration by parts, we get

$$I = \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \, dx + C$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \, dx + C \quad (1)$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} \, dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2}{\sqrt{1-x^2}} \, dx - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{2} \left[ x^2 \sin^{-1} x + \int \sqrt{1-x^2} \, dx - \int \frac{dx}{\sqrt{1-x^2}} \right] \quad (1)$$

$$\Rightarrow I = \frac{1}{2} \left[ x^2 \sin^{-1} x + \frac{x}{2} \cdot \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right] \quad (1)$$

$$\left[ \because \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \right]$$

$$\text{and } \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$= \frac{1}{2} \left[ x^2 \sin^{-1} x + \frac{x}{2} \cdot \sqrt{1-x^2} - \frac{1}{2} \sin^{-1} x \right] + C$$

$$\therefore I = \frac{x^2}{2} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + C \quad (1)$$

**79.** Evaluate  $\int x \cdot \log |(x+1)| \, dx$ .

Delhi 2008C

$$\text{Let } I = \int x \cdot \log |x+1| dx$$

Using integration by parts, we get

$$I = \log |x+1| \int x dx - \int \left[ \frac{d}{dx} \log |x+1| \int x dx \right] dx \quad (1)$$

$$= \log |x+1| \cdot \frac{x^2}{2} - \int \frac{1}{x+1} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2}{2} \log |x+1| - \frac{1}{2} \int \frac{x^2}{x+1} dx$$

$$= \frac{x^2}{2} \log |x+1| - \frac{1}{2} \int \left( x - 1 + \frac{1}{x+1} \right) dx \quad (1\frac{1}{2})$$

$$\left[ \begin{array}{r} x-1 \\ x+1 \overline{) x^2} \\ \underline{x^2+x} \\ -x \\ \underline{-x-1} \\ + \\ \underline{+} \\ 1 \end{array} \right]$$

$$= \frac{x^2}{2} \log |x+1| - \frac{1}{2}$$

$$\left[ \frac{x^2}{2} - x + \log |x+1| \right] + C \left[ \because \int \frac{dx}{x} = \log |x| \right]$$

$$\Rightarrow I = \frac{x^2}{2} \log |x+1| - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2} \log |x+1| + C$$

$$\therefore I = \frac{1}{2} |(x^2 - 1)| \log |x+1| - \frac{x^2}{4} + \frac{x}{2} + C \quad (1\frac{1}{2})$$

### 6 Mark Questions

80. Evaluate  $\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$ .

All India 2014

💡 Firstly, divide numerator and denominator by  $\cos^4 x$  to convert integrand in terms of  $\tan x$  and then put  $\tan x = t$  and convert integrand into standard form which can integrate easily.

$$\text{Let } I = \int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$$

On dividing numerator and denominator by  $\cos^4 x$  in RHS, we get

$$I = \int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx \quad (1)$$

$$= \int \frac{(\sec^2 x)(\sec^2 x)}{\tan^4 x + \tan^2 x + 1} dx \quad (1)$$

Put  $\tan x = t \Rightarrow \sec^2 x dx = dt$  and

$$\sec^2 x = 1 + \tan^2 x = 1 + t^2 \quad (1)$$

$$\therefore I = \int \frac{1+t^2}{t^4 + t^2 + 1} dt \Rightarrow I = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} + 1} dt$$

$$\Rightarrow I = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 3} dt \quad (1)$$

$$\text{Again, put } u = t - \frac{1}{t} \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = du$$

$$\therefore I = \int \frac{du}{u^2 + (\sqrt{3})^2}$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{u}{\sqrt{3}} \right) + C \quad (1)$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{t - \frac{1}{t}}{\sqrt{3}} \right) + C \quad \left[ \because u = t - \frac{1}{t} \right]$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{t^2 - 1}{\sqrt{3}t} \right) + C$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{\tan^2 x - 1}{\sqrt{3} \tan x} \right) + C \quad [\because t = \tan x]$$

(1)

**81.** Evaluate  $\int (\sqrt{\cot x} + \sqrt{\tan x}) dx$ .

All India 2014; Delhi 2010 C

$$\begin{aligned} \text{Let } I &= \int [\sqrt{\cot x} + \sqrt{\tan x}] dx \\ &= \int \sqrt{\tan x} (1 + \cot x) dx \end{aligned}$$

$$\text{Put } \tan x = t^2 \Rightarrow \sec^2 x dx = 2t dt$$

$$\text{or } dx = \frac{2t}{1+t^4} \quad (1)$$

$$[\because 1 + \tan^2 x = \sec^2 x \Rightarrow 1 + t^4 = \sec^2 x]$$

$$\therefore I = \int t \left(1 + \frac{1}{t^2}\right) \frac{2t}{(1+t^4)} dt$$

$$\Rightarrow I = 2 \int \frac{t^2 + 1}{t^4 + 1} dt \quad (1)$$

On dividing numerator and denominator by  $t^2$  in RHS, we get

$$I = 2 \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t^2 + \frac{1}{t^2}\right)} dt = 2 \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} dt \quad (1)$$

$$\text{Again, put } t - \frac{1}{t} = y \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dy$$

$$\therefore I = 2 \int \frac{dy}{y^2 + (\sqrt{2})^2}$$

$$\Rightarrow I = \sqrt{2} \tan^{-1} \frac{y}{\sqrt{2}} + C \quad (1)$$

$$= \sqrt{2} \tan^{-1} \frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + C \quad \left[\because y = t - \frac{1}{t}\right] \quad (1)$$

$$= \sqrt{2} \tan^{-1} \left(\frac{t^2 - 1}{\sqrt{2}t}\right) + C$$

$$= \sqrt{2} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2} \tan x}\right) + C \quad [\because t^2 = \tan x]$$

(1)

82. Evaluate  $\int \frac{1}{\cos^4 x + \sin^4 x} dx$ .

All India 2014

$$\text{Let } I = \int \frac{1}{\cos^4 x + \sin^4 x} dx$$

On dividing numerator and denominator by  $\cos^4 x$  in RHS, we get

$$I = \int \frac{\sec^4 x}{1 + \tan^4 x} dx \Rightarrow I = \int \frac{(\sec^2 x)(\sec^2 x)}{1 + (\tan^2 x)^2} dx$$

$$\Rightarrow I = \int \frac{\sec^2 x (1 + \tan^2 x)}{1 + (\tan^2 x)^2} dx \quad (1)$$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\therefore I = \int \frac{1+t^2}{1+t^4} dt \quad (1)$$

Again, dividing numerator and denominator by  $t^2$  in RHS, we get

$$I = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} + 2 - 2} dt = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2} dt \quad (1)$$

$$\text{Put } t - \frac{1}{t} = u$$

$$\Rightarrow \left(1 + \frac{1}{t^2}\right) dt = du \quad (1)$$

$$\text{Then, } I = \int \frac{du}{u^2 + (\sqrt{2})^2}$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{u}{\sqrt{2}} \right) + C$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{t - \frac{1}{t}}{\sqrt{2}} \right) + C \quad \left[ \because u = t - \frac{1}{t} \right] \quad (1)$$



$$\Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{t^2 - 1}{\sqrt{2}t} \right) + C$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{\tan^2 x - 1}{\sqrt{2} \tan x} \right) + C \quad [ \because t = \tan x ]$$

(1)

**83.** Find  $\int \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx$ .

Delhi 2014C



Firstly, put  $x^2 = t$  and apply partial fraction to convert integrand into some standard form which can integrate easily.

$$\int \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx$$

Let  $x^2 = t$ , then

$$\therefore \frac{t}{(t + 1)(t + 4)} = \frac{A}{t + 1} + \frac{B}{t + 4} \quad \dots(i)$$

$$\Rightarrow t = A(t + 4) + B(t + 1)$$

$$\Rightarrow t = (A + B)t + 4A + B \quad (1)$$

On comparing the coefficients of like powers from both sides, we get

$$A + B = 1, 4A + B = 0 \quad (1)$$

On solving these equations, we get

$$A = -\frac{1}{3} \text{ and } B = \frac{4}{3} \quad (1)$$

From Eq. (i), we get

$$\frac{t}{(t + 1)(t + 4)} = -\frac{1}{3} \frac{1}{t + 1} + \frac{4}{3} \frac{1}{t + 4} \quad (1)$$

$$\begin{aligned} \therefore \int \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx &= -\frac{1}{3} \int \frac{1}{x^2 + 1} dx \\ &+ \frac{4}{3} \int \frac{1}{x^2 + 4} dx \quad [\because x^2 = t] \quad (1) \end{aligned}$$

$$= -\frac{1}{3} \tan^{-1} x + \frac{2}{3} \tan^{-1} \frac{x}{2} + C$$

$$\left[ \because \int \frac{1}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \right] \quad (1)$$

84. Find  $\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx, x \in [0, 1]$ .

All India 2014C

💡 Firstly, use the identity  $\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$  to convert integrand in terms of  $\sin^{-1}$  only. Then, integrate by using substitution.

$$\text{Let } I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$$

We know that,  $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \pi/2$

$$\Rightarrow \cos^{-1}\sqrt{x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x}$$

$$\therefore I = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\pi/2} dx$$

$$\therefore \int \frac{2 \sin^{-1}\sqrt{x} - \frac{\pi}{2}}{\frac{\pi}{2}} dx = \frac{2}{\pi} \int \left(2 \sin^{-1}\sqrt{x} - \frac{\pi}{2}\right) dx$$

$$= \frac{4}{\pi} \int \sin^{-1} \sqrt{x} dx - \int 1 dx = \frac{4}{\pi} \int \sin^{-1} \sqrt{x} dx - x$$

$$\Rightarrow I = \frac{4}{\pi} I_1 - x + C \quad \dots(i) \quad (1)$$

where,  $I_1 = \int \sin^{-1} \sqrt{x} dx$

Put  $\sqrt{x} = t \Rightarrow x = t^2$  and  $dx = 2t dt$

$$I_1 = \int \sin^{-1} t \cdot 2t dt = 2 \int \sin^{-1} t \cdot t dt$$

$$= 2 \left[ \sin^{-1} t \cdot \frac{t^2}{2} - \int \frac{1}{\sqrt{1-t^2}} \cdot \frac{t^2}{2} dt \right] \quad (1)$$

[using integration by parts]

$$= \int \frac{t^2}{\sqrt{1-t^2}} dt = t^2 \sin^{-1} t - \int \frac{(1-t^2) + 1}{\sqrt{1-t^2}} dt$$

$$= t^2 \sin^{-1} t + \int \sqrt{1-t^2} dt - \int \frac{1}{\sqrt{1-t^2}} dt \quad (1)$$

$$= t^2 \sin^{-1} t + \frac{t\sqrt{1-t^2}}{2} + \frac{1}{2} \sin^{-1} t - \sin^{-1} t$$

$$\left( t^2 - \frac{1}{2} \right) \sin^{-1} t + \frac{1}{2} t \sqrt{1-t^2} \quad (1)$$

$$= \frac{1}{2} [(2x-1) \sin^{-1} \sqrt{x} + \sqrt{x}\sqrt{1-x}]$$

$$= \frac{1}{2} [(2x-1) \sin^{-1} \sqrt{x} + \sqrt{x-x^2}] \quad (1)$$

On putting the value of  $I_1$  in Eq. (i), we get

$$\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$$

$$= \frac{2}{\pi} [(2x-1) \sin^{-1} \sqrt{x} + \sqrt{x-x^2}] - x + C \quad (1)$$

**85.** Find  $\int \frac{x^2 + x + 1}{(x+1)^2(x+2)} dx$ .

Delhi 2014C

$$\int \frac{x^2 + x + 1}{(x + 1)^2(x + 2)} dx$$

The integrand  $\frac{x^2 + x + 1}{(x + 1)^2(x + 2)}$  is a proper rational function.

$$\therefore \frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{x + 2} \dots (i)$$

(1)

$$\Rightarrow x^2 + x + 1 = A(x + 1)(x + 2) + B(x + 2) + C(x + 1)^2$$

$$\Rightarrow x^2 + x + 1 = A(x^2 + 3x + 2) + B(x + 2) + C(x^2 + 2x + 1)^2$$

$$\Rightarrow x^2 + x + 1 = (A + C)x^2 + (3A + B + 2C)x + (2A + 2B + C) \quad (1)$$

On comparing the coefficients of like powers from both sides, we get (1)

$$A + C = 1, 3A + B + 2C = 1 \text{ and } 2A + 2B + C = 1$$

On solving these equations, we get (1)

$$A = -2, B = 1 \text{ and } C = 3$$

From Eq. (i), we get

$$\frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{-2}{x + 1} + \frac{1}{(x + 1)^2} + \frac{3}{x + 2} \quad (1)$$

$$\begin{aligned} \therefore \int \frac{x^2 + x + 1}{(x + 1)^2(x + 2)} dx &= -2 \int \frac{1}{x + 1} dx \\ &\quad + \int \frac{dx}{(x + 1)^2} + 3 \int \frac{dx}{x + 2} \\ &= -2 \log|x + 1| - \frac{1}{x + 1} + 3 \log|x + 2| + C \quad (1) \end{aligned}$$

**86.** Find  $\int \frac{\sqrt{x^2 + 1}(\log(x^2 + 1) - 2 \log x)}{x^4} dx.$  All India 2014C

$$\begin{aligned} \text{Let } I &= \int \frac{\sqrt{x^2+1} [\log(x^2+1) - 2 \log x]}{x^4} dx \\ &= \int \frac{\sqrt{x^2+1} \log\left(\frac{x^2+1}{x^2}\right)}{x^4} dx \\ &\quad \left[ \because \log m - a \log n = \log \frac{m}{n^a} \right] \quad (1) \end{aligned}$$

$$= \int \frac{x \sqrt{1 + \frac{1}{x^2}} \log\left(1 + \frac{1}{x^2}\right)}{x^4} dx \quad (1)$$

$$= \int \frac{\sqrt{1 + \frac{1}{x^2}} \log\left(1 + \frac{1}{x^2}\right)}{x^3} dx$$

$$\text{Put } 1 + \frac{1}{x^2} = t, \text{ then } \frac{-2}{x^3} dx = dt \Rightarrow \frac{dx}{x^3} = -\frac{dt}{2} \quad (1)$$

$$\therefore I = -\frac{1}{2} \int \sqrt{t} \log t \, dt$$

$$= -\frac{1}{2} \left[ \log t \times \frac{t^{3/2}}{3/2} - \int \frac{t^{3/2}}{3/2} \times \frac{1}{t} dt \right]$$

[using integration by parts] (1)

$$= -\frac{1}{3} [t^{3/2} \log t - \int \sqrt{t} dt]$$

$$= -\frac{1}{3} \left[ t^{3/2} \log t - \frac{t^{3/2}}{3/2} \right] + C \quad (1)$$


$$= -\frac{1}{3} t^{3/2} \left[ \log t - \frac{2}{3} \right] + C$$

$$= -\frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{3/2} \left[ \log\left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right] + C$$

$$\left[ \because t = 1 + \frac{1}{x^2} \right] \quad (1)$$

87. Evaluate  $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ .

Delhi 2012

•  Firstly, put  $x = \sin t$  and then use integration by parts and simplify it.

Let  $I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Put  $\sin^{-1} x = t \Rightarrow x = \sin t$

$\Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$  (1½)

$\therefore I = \int t \sin t dt$

Using integration by parts, taking  $t$  as the first function and  $\sin t$  as the second function, we get

$I = t \int \sin t dt - \int \left[ \frac{d}{dt}(t) \cdot \int \sin t dt \right] dt$  (1½)

$\Rightarrow I = -t \cos t - \int (1 \times -\cos t) dt$   
 $= -t \cos t + \int \cos t dt$

$\Rightarrow I = -t \cos t + \sin t + C$  (1½)

$\Rightarrow I = -t \sqrt{1-\sin^2 t} + \sin t + C$

$[\because \cos^2 t = 1 - \sin^2 t \Rightarrow \cos t = \sqrt{1 - \sin^2 t}]$

$\therefore I = -\sin^{-1} x \sqrt{1-x^2} + x + C$

$[\because t = \sin^{-1} x \text{ and } x = \sin t]$  (1½)

88. Evaluate  $\int \frac{x^2 + 1}{(x-1)^2(x+3)} dx$ .

Delhi 2012



Using partial fraction, such that

$$\frac{1}{(x-a)^2(x+b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x+b}$$

and then integrate it to get the desired result.

Let 
$$I = \int \frac{x^2 + 1}{(x-1)^2(x+3)} dx$$

Again, let

$$\frac{x^2 + 1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3} \dots(i) \quad (1)$$

$$\Rightarrow \frac{x^2 + 1}{(x-1)^2(x+3)}$$

$$= \frac{A(x-1)(x+3) + B(x+3) + C(x-1)^2}{(x-1)^2(x+3)}$$

$$\Rightarrow x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$$

$$\Rightarrow x^2 + 1 = A(x^2 + 2x - 3) + B(x+3)$$

$$+ C(x^2 + 1 - 2x)$$

$$\Rightarrow x^2 + 1 = (A+C)x^2 + (2A+B-2C)x$$

$$+ (-3A+3B+C)$$

On comparing the coefficients of  $x^2$ ,  $x$  and constant terms from both sides, we get

$$A + C = 1 \quad \dots(ii)$$

$$2A + B - 2C = 0 \quad \dots(iii)$$

$$-3A + 3B + C = 1 \quad \dots(iv) \quad (1)$$

On multiplying Eq. (iii) by 3 and subtracting it from Eq. (iv), we get

$$-9A + 7C = 1 \quad \dots(v)$$

On multiplying Eq. (ii) by 7 and subtracting it from Eq. (v), we get

$$-16A = -6 \quad \dots(vi)$$

$$\therefore A = \frac{6}{16} = \frac{3}{8} \quad (1)$$

On putting  $A = \frac{3}{8}$  in Eq. (ii), we get

$$3 + C = 1 \Rightarrow C = 1 - \frac{3}{8} = \frac{5}{8}$$





$$\frac{3}{8} + C = 1 \Rightarrow C = 1 - \frac{3}{8} = \frac{5}{8}$$

On putting  $A = \frac{3}{8}$  and  $C = \frac{5}{8}$  in Eq. (iii), we get

$$\frac{3}{4} + B - \frac{5}{4} = 0 \Rightarrow B - \frac{2}{4} = 0 \Rightarrow B = \frac{2}{4} = \frac{1}{2}$$

Thus,  $A = \frac{3}{8}, B = \frac{1}{2}$  and  $C = \frac{5}{8}$  (1)

∴ Eq. (i) becomes

$$\frac{x^2 + 1}{(x-1)^2(x+3)} = \frac{3/8}{x-1} + \frac{1/2}{(x-1)^2} + \frac{5/8}{x+3}$$

On integrating both sides, we get

$$I = \int \frac{x^2 + 1}{(x-1)^2(x+3)} dx = \frac{3}{8} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{(x-1)^2} + \frac{5}{8} \int \frac{dx}{x+3}$$

$$= \frac{3}{8} \log|x-1| + \frac{1}{2} \left( \frac{-1}{x-1} \right) + \frac{5}{8} \log|x+3| + C$$

Hence,  $I = \frac{3}{8} \log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8} \log|x+3| + C$  (1)

**89.** Evaluate  $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}} dx$ . All India 2011

💡 If the integral is of the form  $\int \frac{gx+d}{\sqrt{ax^2+bx+c}} dx$ , then we take  $gx+d = A \cdot \frac{d}{dx}(ax^2+bx+c) + B$  and then integrate it.

$$\text{Let } I = \int \frac{(6x+7) dx}{\sqrt{(x-5)(x-4)}} \Rightarrow I = \int \frac{(6x+7) dx}{\sqrt{x^2-9x+20}}$$

Here,  $(6x+7)$  can be written as

$$6x+7 = A \cdot \frac{d}{dx}(x^2-9x+20) + B$$

$$\Rightarrow 6x+7 = A(2x-9) + B \quad \text{(i) (1)}$$

On comparing the coefficients of  $x$  and constant terms from both sides, we get

$$2A = 6 \Rightarrow A = 3 \text{ and } -9A + B = 7$$

$$\Rightarrow -9(3) + B = 7 \quad [ \because A = 3 ]$$

$$\Rightarrow -27 + B = 7$$

$$\therefore B = 34 \quad \text{(1)}$$

On putting the values of  $A$  and  $B$  in Eq. (i), we get  $6x + 7 = 3(2x - 9) + 34$

$\therefore$  Given integral can be written as

$$I = \int \frac{3(2x - 9) + 34}{\sqrt{x^2 - 9x + 20}} dx$$

$$\Rightarrow I = 3 \int \frac{2x - 9}{\sqrt{x^2 - 9x + 20}} dx + 34 \int \frac{dx}{\sqrt{x^2 - 9x + 20}}$$

$$\Rightarrow I = 3I_1 + 34I_2 \quad \dots\text{(ii) (1)}$$

$$\text{where, } I_1 = \int \frac{2x - 9}{\sqrt{x^2 - 9x + 20}} dx$$

$$\text{Put } x^2 - 9x + 20 = t \Rightarrow (2x - 9) dx = dt$$

$$\therefore I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} dt = 2t^{1/2} = 2\sqrt{t}$$

$$= 2\sqrt{x^2 - 9x + 20} \quad \dots\text{(iii) (1)}$$

$$\text{and } I_2 = \int \frac{dx}{\sqrt{x^2 - 9x + 20}}$$

$$\Rightarrow I_2 = \int \frac{dx}{\sqrt{x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}}}$$

$$\Rightarrow I_2 = \int \frac{dx}{\sqrt{\left(x - \frac{9}{2}\right)^2 + \left(20 - \frac{81}{4}\right)}}$$

$$\Rightarrow I_2 = \int \frac{dx}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}}$$

$$\Rightarrow I_2 = \int \frac{dx}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$\therefore I_2 = \log \left| \left(x - \frac{9}{2}\right) + \sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| \dots \text{(iv)}$$

$$\left[ \because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log |x + \sqrt{x^2 - a^2}| \right] \text{(1)}$$

On putting the values of  $I_1$  and  $I_2$  from Eqs. (iii) and (iv) in Eq. (ii), we get


$$I = 3(2\sqrt{x^2 - 9x + 20})$$

$$+ 34 \left[ \log \left| \left(x - \frac{9}{2}\right) + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}} \right| \right] + C$$

$$\therefore I = 6\sqrt{x^2 - 9x + 20}$$

$$+ 34 \log \left| x - \frac{9}{2} + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}} \right| + C \text{ (1)}$$

**90.** Evaluate  $\int \frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} dx$ . All India 2009C

 Firstly, use the method of partial fraction and then integrate it.

Let 
$$I = \int \frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} dx$$

Again, let 
$$\frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + 1} \quad (1)$$
$$= \frac{A(x^2 + 1) + (Bx + C)(x + 2)}{(x + 2)(x^2 + 1)}$$

$$\Rightarrow x^2 + x + 1 = A(x^2 + 1) + (Bx + C)(x + 2) \quad \dots(i)$$

$$\Rightarrow x^2 + x + 1 = (A + B)x^2 + (C + 2B)x + (A + 2C)$$

On putting  $x = -2$  in Eq. (i), we get

$$4 - 2 + 1 = A(4 + 1) + 0 \Rightarrow 3 = 5A \Rightarrow A = 3/5$$

On equating the coefficients of  $x^2$  and constant terms, we get

$$A + B = 1 \quad \dots(ii)$$

$$A + 2C = 1 \quad \dots(iii)$$

On putting  $A = \frac{3}{5}$  in Eq. (ii), we get

$$\frac{3}{5} + B = 1 \Rightarrow B = 1 - \frac{3}{5} = \frac{2}{5}$$

On putting  $A = \frac{3}{5}$  in Eq. (iii), we get

$$\frac{3}{5} + 2C = 1 \Rightarrow 2C = 1 - \frac{3}{5}$$

$$\Rightarrow 2C = 2/5 \Rightarrow C = 1/5 \quad (2)$$

$$\therefore I = \int \frac{3/5}{x+2} dx + \int \frac{\frac{2}{5}x + \frac{1}{5}}{x^2 + 1} dx \quad (1)$$

$$= \frac{3}{5} \int \frac{dx}{x+2} + \frac{2}{5} \int \frac{xdx}{x^2 + 1} + \frac{1}{5} \int \frac{dx}{x^2 + 1}$$

Put  $x^2 + 1 = t \Rightarrow 2x dx = dt$

$$\therefore I = \frac{3}{5} \int \frac{dx}{x+2} + \frac{1}{5} \int \frac{dt}{t} + \frac{1}{5} \int \frac{dx}{(x)^2 + (1)^2} \quad (1)$$

$$= \frac{3}{5} \log|x+2| + \frac{1}{5} \log|t| + \frac{1}{5} \cdot \frac{1}{1} \tan^{-1} \frac{x}{1} + C$$

$$\left[ \because \int \frac{dx}{x} = \log|x| \text{ and } \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \right]$$

$$= \frac{3}{5} \log|x+2| + \frac{1}{5} \log|x^2 + 1|$$

$$+ \frac{1}{5} \tan^{-1} x + C \quad (1)$$

**91.** Evaluate  $\int \frac{\tan x + \tan^3 x}{1 + \tan^3 x} dx$ . HOTS; Delhi 2009C



Firstly, simplify the given integrand in initial level and then apply the method of partial fraction after that integrate it and get the desired result.

$$\begin{aligned} \text{Let } I &= \int \frac{\tan x + \tan^3 x}{1 + \tan^3 x} dx \\ &= \int \frac{\tan x(1 + \tan^2 x)}{1 + \tan^3 x} dx = \int \frac{\tan x \sec^2 x}{1 + \tan^3 x} dx \\ &\quad [\because 1 + \tan^2 x = \sec^2 x] \quad (1) \end{aligned}$$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{t dt}{1 + t^3} \\ &\quad [\because (a^3 + b^3) = (a + b)(a^2 - ab + b^2)] \\ &= \int \frac{t dt}{(1 + t)(1 + t^2 - t)} \quad (1) \end{aligned}$$

$$\text{Let } \frac{t}{(1 + t)(1 + t^2 - t)} = \frac{A}{1 + t} + \frac{Bt + C}{t^2 - t + 1}$$

$$\Rightarrow t = A(t^2 - t + 1) + (Bt + C)(1 + t)$$

$$\Rightarrow t = (A + B)t^2 + (-A + B + C)t + (A + C)$$

On putting  $t = -1$ , we get

$$-1 = A(1 + 1 + 1) + 0$$

$$\Rightarrow A = \frac{-1}{3}$$

On equating the coefficients of  $t^2$  and constant terms from both sides, we get  $A + B = 0$

$$\Rightarrow \frac{-1}{3} + B = 0 \Rightarrow B = \frac{1}{3} \text{ and } A + C = 0$$

$$\Rightarrow -\frac{1}{3} + C = 0 \Rightarrow C = \frac{1}{3} \quad (1)$$

$$\begin{aligned} \therefore I &= \int \frac{(-1/3)}{1+t} dt + \int \frac{\frac{1}{3}t + \frac{1}{3}}{t^2 - t + 1} dt \\ &= -\frac{1}{3} \int \frac{dt}{1+t} + \frac{1}{3} \int \frac{t+1}{t^2 - t + 1} dt \end{aligned}$$



$$\begin{aligned}
&= -\frac{1}{3} \int \frac{dt}{1+t} + \frac{1}{6} \int \frac{2t-1+3}{t^2-t+1} dt \\
&= -\frac{1}{3} \int \frac{dt}{1+t} + \frac{1}{6} \int \frac{2t-1}{t^2-t+1} dt \\
&\quad + \frac{3}{6} \int \frac{dt}{t^2-t+1 + \frac{1}{4} - \frac{1}{4}} \quad (1)
\end{aligned}$$

Let  $z = t^2 - t + 1$

$\Rightarrow dz = (2t - 1) dt$  in middle integral, we get

$$I = -\frac{1}{3} \int \frac{dt}{1+t} + \frac{1}{6} \int \frac{1}{z} dz + \frac{1}{2} \int \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$\begin{aligned}
\Rightarrow I &= -\frac{1}{3} \log|1+t| + \frac{1}{6} \log z \\
&\quad + \frac{1}{2} \int \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \quad (1)
\end{aligned}$$

$$= -\frac{1}{3} \log|1 + \tan x| + \frac{1}{6} \log(t^2 - t + 1)$$

$$+ \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left( \frac{t - \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C \quad [\because t = \tan x]$$

$$= -\frac{1}{3} \log|1 + \tan x| + \frac{1}{6} \log|\tan^2 x - \tan x + 1|$$

$$+ \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2 \tan x - 1}{\sqrt{3}} \right) + C \quad (1)$$

**92.** Evaluate  $\int \frac{x^2}{x^4 + x^2 + 1} dx$ . **HOTS; Delhi 2008C**

Let  $I = \int \frac{x^2}{x^4 + x^2 + 1} dx = \frac{1}{2} \int \frac{2x^2}{x^4 + x^2 + 1} dx$

$$\Rightarrow I = \frac{1}{2} \int \frac{x^2 + 1 + x^2 - 1}{x^4 + x^2 + 1} dx$$

$$\Rightarrow I = \frac{1}{2} \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx + \frac{1}{2} \int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$$

$$\Rightarrow I = \frac{1}{2} I_1 + \frac{1}{2} I_2 \quad \dots(i) \quad (1)$$

where  $I_1 = \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx$

On dividing numerator and denominator by  $x^2$ , we get

$$I_1 = \int \frac{\left(\frac{x^2}{x^2} + \frac{1}{x^2}\right)}{\left(\frac{x^4}{x^2} + \frac{x^2}{x^2} + \frac{1}{x^2}\right)} dx \quad (1)$$

$$\Rightarrow I_1 = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + 1 + \frac{1}{x^2}\right)} dx$$

$$\Rightarrow I_1 = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 2 + 1} dx$$

$$\Rightarrow I_1 = \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{3})^2} dx \quad (1)$$

Put  $x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$

$$\therefore I_1 = \int \frac{dt}{t^2 + (\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} \quad \left[ \because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right) \right]$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{\left(x - \frac{1}{x}\right)}{\sqrt{3}} = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{x\sqrt{3}}\right) \quad (1)$$

and  $I_2 = \int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$



$$= x^2 + x^2 + 1$$

On dividing numerator and denominator by  $x^2$ , we get

$$\begin{aligned} I &= \int \frac{\left(\frac{x^2}{x^2} - \frac{1}{x^2}\right)}{\left(\frac{x^4}{x^2} + \frac{x^2}{x^2} + \frac{1}{x^2}\right)} dx = \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x^2 + 1 + \frac{1}{x^2}\right)} dx \\ &= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 2 + 1} dx = \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 1} dx \end{aligned}$$

Put  $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$

$$\therefore I_2 = \int \frac{dt}{t^2 - 1} = \int \frac{dt}{t^2 - (1)^2}$$

$$\left[ \because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \right]$$

$$= \frac{1}{2 \times 1} \log \left| \frac{t-1}{t+1} \right| \quad (1)$$

$$= \frac{1}{2} \log \left| \frac{\left(x + \frac{1}{x}\right) - 1}{\left(x + \frac{1}{x}\right) + 1} \right| \quad \left[ \because t = x + \frac{1}{x} \right]$$

$$= \frac{1}{2} \log \left| \frac{x^2 + 1 - x}{x^2 + 1 + x} \right|$$

Now, on putting the values of  $I_1$  and  $I_2$  in Eq.(i), we get

$$\begin{aligned} I &= \frac{1}{2} \cdot \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{x^2 - 1}{x\sqrt{3}} \right) \\ &\quad + \frac{1}{2} \cdot \frac{1}{2} \log \left| \frac{x^2 - x + 1}{x^2 + x + 1} \right| + C \end{aligned}$$

$$\therefore I = \frac{1}{2\sqrt{3}} \tan^{-1} \left( \frac{x-1}{x\sqrt{3}} \right) + \frac{1}{4} \log \left| \frac{x^2 - x + 1}{x^2 + x + 1} \right| + C \quad (1)$$

**Note** In this type of integral, we cannot use the method of partial fraction directly.